Game Pricing and Double Sequence of Random Variables

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Abstract

In this paper, we study a random payoff with positive or plus infinite expectation and determine the optimal proportion of investment for maximizing the limit expectation of growth rate per attempt. With this objective, we introduce a new pricing method in which the price is different from that obtained by the Black-Scholes formula for a European option. We will price the St. Petersburg game nicely.

Keywords

Proportion of Investment; Pricing of Random Payoff; Black-Scholes Formula

Introduction

The portfolio pricing equation (Luenberger 1998) is useful for determining prices of securities only if the optimal portfolio has been already known. In this paper, we determine both the price and optimal proportion of investment for a random payoff.

The determination of the utility function is more experimental than mathematical. With an additional reason that $X = \exp(\log X)$ frequently provides $E[X] \neq \exp(E[\log X])$, the log utility approach (Luenberger 1998) is not preferable.

The investor should repeatedly invest a fixed proportion of his or her own current capital without borrowing. As a rule, if the investor invests 1 dollar, then he or she receives $a(x)$ dollars (including the invested 1 dollar) with a cumulative distribution function $F(x)$ defined on an interval I. For simplicity, we omit the currency notation. Let $M > 0$ be the investor's capital, $u > 0$ the price of the random payoff $(a(x),\quad F(x))$, and $0 \le t \le 1$ the proportion of investment. Then, after one attempt, he or she has capital of $Mta(x)/u + M(1-t)$ if x occurs. It should be noted that the reserved part $M(1-t)$ does not include the interest, which is the custom, for example, in foreign exchange accounts.

Let $M_n > 0$ be the capital after n attempts. In general, growth rate implies $M_{n+1}/M_n - 1$ or $\log(M_{n+1}/M_n)$ after one attempt. However, for the purposes of succinctness in this paper M_{n+1}/M_n is used to define the growth rate. In this context, the growth rate per attempt is defined as $(M_n/M_0)^{\frac{1}{n}}$.

Without dealing with $(M_n/M_0)^{\frac{1}{n}}$ directly, this paper defines a double sequence of random variables $\{X_{N,n}\}$ with respect to the bounded step functions $\{f_N(x)\}$ such that $\lim_{N\to+\infty} f_N(x) = a(x)$. It is shown that the finite limit $\lim_{\substack{n\to+\infty \\ N\to+\infty}} E[X_{N,n}]$ exists if, and only if, the random payoff is effective. In this case, the equalities

$$\lim_{\substack{n\to+\infty \\ N\to+\infty}} E[X_{N,n}] = G_u(t) := \exp\left(\int_I \log(a(x)t/u - t + 1)dF(x)\right)$$
$$= \exp(E[\log(a(x)t/u - t + 1)]) \text{ and } \lim_{\substack{n\to+\infty \\ N\to+\infty}} V[X_{N,n}] = 0$$

are obtained. These equalities again support the well-known assertion that although in principle an investor may choose any utility function, a repetitive situation tends to hammer the utility into one that is close to the logarithm (Kelly 1956, Luenberger 1998).

We study the optimal proportion of investment t_u, for the price $u > 0$ in order to maximize the limit expectation of growth rate per attempt. In order to determine the price of the random payoff, we require a risk-free interest rate, $r > 0$, for a particular period. The equation $G_u(t_u) = r + 1$ (if r is simple) or $G_u(t_u) = e^r$ (if r is continuously compounded) is used to determine the price of a random payoff. If $a(x) \ge 0$ for each $x \in I$, then the existence and uniqueness of the price are guaranteed by the fact that $G_u(t_u)$ is continuous and strictly decreases from $+\infty$ to 1 with respect to $0 < u < E := \int_I a(x)dF(x)$ (Theorem 4.1). In this context, the price of the St. Petersburg game

(Bernoulli 1954) is determined to be 5.0815 if the risk-free interest rate is 4% (Example 6.4). On the other hand, the Black-Scholes formula is deduced from the equation $E/u = e^r$, where E is the expectation of a European option (Example 6.6).

Any random variable $X:(\Omega, \quad P) \to R$ reduces a random payoff $(x, \quad F(x))$, where $F(x) \quad := P(\{\omega \in \Omega \mid X(\omega) \le x\})$.

Optimal Proportion of Investment

Assume that the payoff function $a(x)$ is measurable with the distribution function $F(x)$ defined on an interval $I \subseteq (-\infty \ , \ +\infty)$. Set $\xi := \inf_{x \in I} a(x)$. We also assume that $\xi > -\infty$ and ξ is the essential infimum of $a(x)$, that is, $\int_{a(x)<\xi+\varepsilon} dF(x) \ > 0$ for each $\varepsilon > 0$. Further, assume that $a(x)$ is not a constant function (a.e.), that is, $\int_{a(x)<\xi+\delta} dF(x) < 1$ for some $\delta > 0$.

We use the following notation with the Lebesgue-Stieltjes integral.

$$E := \int_I a(x) dF(x), \quad H := \int_I \frac{1}{a(x)} dF(x),$$

$$H_\xi := \int_I \frac{1}{a(x)-\xi} dF(x). \tag{1}$$

In this paper, we assume that $E > 0$. If $\int_{a(x)=\xi} dF(x) > 0$, we define $H_\xi \ = +\infty$ and $1/H_\xi \ = 0$.

Since $a(x)$ is not constant, we have $\xi < E$, $H_\xi > 0$, and $1/H_\xi < +\infty$. From the relation

$$1 = \left(\int_I \sqrt{a(x)-\xi} \times \frac{1}{\sqrt{a(x)-\xi}} dF(x) \right)^2$$

$$< \int_I (a(x)-\xi) dF(x) \times \int_I \frac{1}{a(x)-\xi} dF(x) = (E-\xi) H_\xi,$$

we have $\xi + 1/H_\xi < E$. In particular, if $\xi = 0$, $0 \le 1/H < E$. If $\xi > 0$, then using $1/\xi \ge 1/a(x)$ and $1 = \sqrt{a(x)} \times (1/\sqrt{a(x)})$, we have $\xi < 1/H < E$.

For price $u > 0$, let $t_u \in [0, \quad 1]$ be the optimal proportion of investment. The precise definition of the term "optimal" and its significance are shown beneath Lemma 5.1. Here, we present certain properties of t_u in order to explain the approximate outline of the paper.

(**a**) If $u > E, \ t_u = 0$.

Assume that $u > E$ and $t \in (0 \ , \ 1]$, then the expectation of payoffs,

$$Mt \int_I a(x)/u \quad dF(x) + M(1-t) = M - M(1-E/u)t,$$

is less than M. More precisely, using Jensen's inequality, we have $G_u(t) < 1-(1-E/u)t < 1 = G_u(0)$ for each $t \in (0 \ , \ 1]$. Therefore, $t_u = 0$.

In the proof of Theorem 5.1, we will show that:

$$\{u \mid t_u = 0\} = \begin{cases} [E, +\infty), & \text{if } E < +\infty, \\ \phi, & \text{if } E = +\infty. \end{cases} \tag{2}$$

(**b**) If $\xi > 0$ and $0 < u \le \xi$, then $t_u = 1$.

From $0 < u \le \xi \le a(x)$ and $t \in [0 \ , \ 1)$, after one attempt, we have $Mta(x)/u + M(1-t) \quad = Ma(x)/u \quad -M(1-t)(a(x)/u-1) \quad \le Ma(x)/u$ for each $x \in I$. This implies that $G_u(t) < G_u(1)$ for each $t \in [0 \ , \ 1)$, that is, $t_u = 1$.

Accordingly, in the proof of Theorem 5.1, we will also show that

$$\{u \mid t_u = 1\} = \begin{cases} (0, 1/H], & \text{if } \xi > 0, \text{or } \xi = 0 \text{ and } H < +\infty, \\ \phi, & \text{if } \xi < 0, \text{or } \xi = 0 \text{ and } H = +\infty, \end{cases} \tag{3}$$

which yields a maximum price of $1/H$ at which all the capital should be repeatedly invested.

(**c**) If $\max(0,\xi) < u$, $t_u \le u/(u-\xi)$.

If $t > u/(u-\xi)$, $u-\xi-u/t > 0$. Therefore, the negative result $Mta(x)/u \quad +M(1-t) < 0$ occurs with a positive probability $\int_{\xi \le a(x) < \xi+(u-\xi-u/t)} dF(x) > 0$. This contradicts the concept of continual investment without borrowing.

In the proof of Theorem 5.1, the condition for $t_u = u/(u-\xi)$ is shown such that:

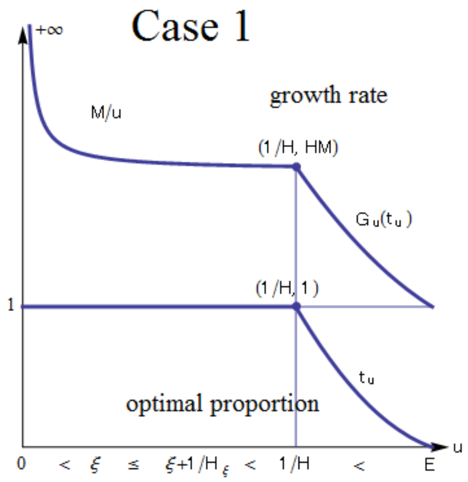
$$\{u \mid t_u = \frac{u}{u-\xi}\} = \begin{cases} (0, \xi + 1/H_\xi], & \text{if } \xi \le 0 \text{ and } \xi + 1/H_\xi > 0, \\ \phi, & \text{if } \xi > 0 \text{ or } \xi + 1/H_\xi \le 0. \end{cases} \tag{4}$$

(**d**) Theorem 5.1 also shows that $t_u \notin \{0, \ 1, \ u/(u-\xi)\}$ if and only if $1/H \quad < u < E$ (if $\xi \ge 0$) or $\max(0 \ , \ \xi + 1/H_\xi) < u < E$ (if $\xi < 0$). In this case, t_u can be uniquely determined by the property:

$$\int_I \frac{a(x)-u}{a(x)t_u - ut_u + u} dF(x) = 0. \tag{5}$$

Here, we offer 5 graphics to facilitate reader's understanding. Recall the definitions: $\xi := \inf_{x \in I} a(x)$, $\quad E := \int_I a(x) dF(x)$, $\quad H := \int_I \frac{1}{a(x)} dF(x)$, and $H_\xi := \int_I \frac{1}{a(x)-\xi} dF(x)$.

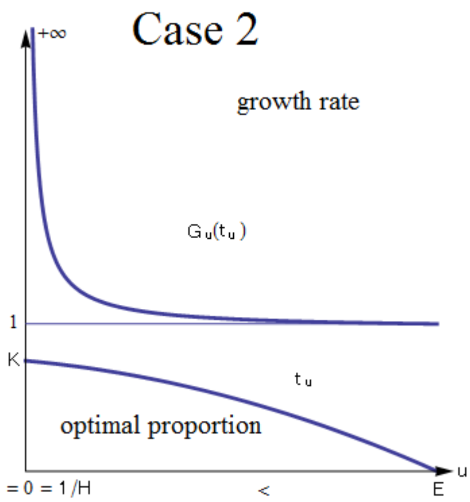
Case 1: $\xi > 0$.



$M := \exp(\int_I \log a(x)dF(x))$ and

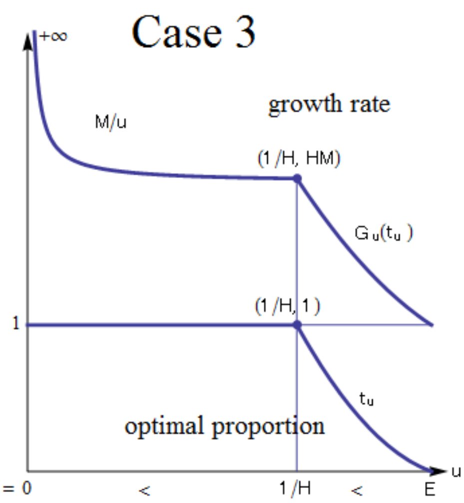
$G_u(t) := \exp(\int_I \log(a(x)t/u - t + 1)dF(x))$.

Case 2: $\xi = 0$ and $H = +\infty$.

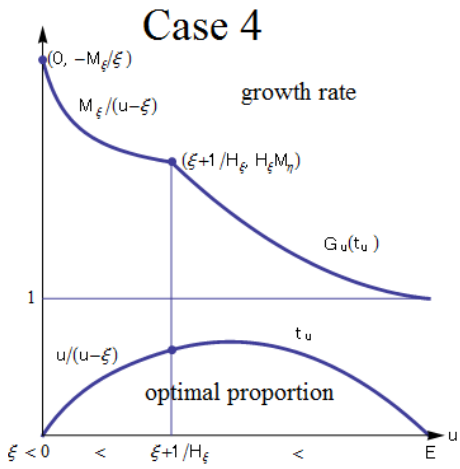


$K := \int_{a(x)>0} dF(x)$.

Case 3: $\xi = 0$ and $H < +\infty$.

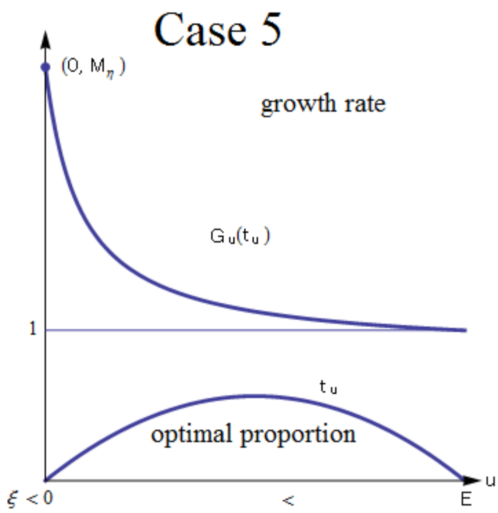


Case 4: $\xi < 0$ and $\xi + 1/H_\xi > 0$.



$M_\xi := \exp(\int_I \log(a(x) - \xi)dF(x))$.

Case 5: $\xi < 0$ and $\xi + 1/H_\xi \le 0$.



$\eta := \lim_{u \to 0^+} t_u / u$ and $M_\eta := \exp(\int_I \log(a(x)\eta + 1)dF(x))$.

Pre-Optimal Proportion

We denote the integral $\int_I (a(x) - \beta)/(a(x)z - z\beta + \beta)dF(x)$ by $w_\beta(z)$, in which z and β are complex variables.

Lemma 3.1. *The function* $w_\beta(z)$ *is holomorphic with respect to two complex variables* $z := t + si$ *and* $\beta := u + hi$ *such that,*

(a) $\max(\varepsilon, \xi) < u < L < +\infty$,

(b) $|h| < \varepsilon^6 / (32(L+1)R^2)$,

(c) $|z| < R$ *and* $z \notin \{|s| \le \varepsilon\} \cap \{t \le \varepsilon$ *or* $t \ge u/(u-\xi) - \varepsilon\}$,

where $0 < \varepsilon < \min(1/2, u/(2(u-\xi)))$, $\max(2, u/(u-\xi)) < R < +\infty$, $i := \sqrt{-1}$, $\mathrm{Im}(z) = s$ *and* $\mathrm{Im}(\beta) = h$.

Proof. We obtain certain operator exchange properties such as

$$\frac{\partial}{\partial t} w_\beta(z) = \int_I \frac{\partial}{\partial t}\left(\frac{a(x)-\beta}{a(x)z - z\beta + \beta}\right) dF(x)$$

by proving that the related integrands are bounded. Because $(a(x)-\beta)/(a(x)z -z\beta+\beta)$ satisfies the Cauchy-Riemann equations, $w_\beta(z)$ is shown to be holomorphic due to Hartogs's theorem.

It should be noted that the condition (a) above leads to $\beta \neq 0$, and if $a(x) \neq \beta$, then we have

$$\frac{a(x)-\beta}{a(x)z - z\beta + \beta} = \frac{1}{z - \frac{1}{1-\frac{a(x)}{\beta}}}.$$

The partial derivatives of $(a(x)-\beta)/(a(x)z -z\beta+\beta)$ with respect to t, s, u, and h are 0, 0, -1 and $-i$, respectively, at $\{x \mid a(x)=\beta\}$. In the following four cases, we assume that $a(x) \neq \beta$.

In this proof, we will frequently use the inequality $|1/(1-z)| \le |2/z|$ if $|z| \ge 2$.

$<$Case 1$>$ $|a(x)| \ge 8(L+1)/\varepsilon$.

As a result of the conditions, we have $|a(x)/u| > |a(x)/L| > |a(x)/(L+1)| \ge 8/\varepsilon > 16$, which leads to $|1/(1-a(x)/u)| \le |2/(a(x)/u))| < |2(L+1)/a(x)| \le \varepsilon/4$. On the other hand, the inequality $|a(x)/\beta| > \left|\frac{a(x)}{L+1}\right| \ge 8/\varepsilon > 16$ leads to $|1/(1-a(x)/\beta)| \le |2/(a(x)/\beta)| < |2(L+1)/a(x)| \le \varepsilon/4$, where $|\beta| \le u + |h| < L+1$.

Moreover, from $\xi \le a(x)$ we have $1 - a(x)/u \le (u-\xi)/u$. If $1-a(x)/u > 0$ then $u/(u-\xi) \le 1/(1-a(x)/u)$, which leads to $|z - 1/(1-a(x)/u))| > \varepsilon$ due to (c). If $1-a(x)/u < 0$ then $1/(1-a(x)/u) < 0$, which leads to $|z-1/(1-a(x)/u))| > \varepsilon$ due to (c). If $1-a(x)/u = 0$ then $L > u = |a(x)| \ge 8(L+1)/\varepsilon$, which is a contradiction. Therefore, we have

$$\left| z - \frac{1}{1-\frac{a(x)}{\beta}} \right| = \left| z - \frac{1}{1-\frac{a(x)}{u}} + \frac{1}{1-\frac{a(x)}{u}} - \frac{1}{1-\frac{a(x)}{\beta}} \right| > \frac{\varepsilon}{2},$$

which establishes $\dfrac{1}{\left| z - \frac{1}{1-\frac{a(x)}{\beta}} \right|} < \dfrac{2}{\varepsilon}$.

Moreover, using $|a(x)/(1-a(x)/\beta)| \le |2\beta|$, $|1/(1-a(x)/\beta)| < \varepsilon/4$, and $1/|\beta| < 1/\varepsilon$, we have

$$\left| \frac{a(x)}{\beta^2\left(z - \frac{1}{1-\frac{a(x)}{\beta}}\right)^2 \left(1-\frac{a(x)}{\beta}\right)^2} \right| < \frac{2}{\varepsilon^2}.$$

$<$Case 2$>$ $|a(x)| < 8(L+1)/\varepsilon$ and $|a(x)/\beta - 1| \le \varepsilon/R$.

Since $|(a(x)/\beta - 1)z| \le \varepsilon$, we have $|a(x)z/\beta - z + 1| \ge 1 - \varepsilon$. Therefore,

$$\left| \frac{\frac{a(x)}{\beta} - 1}{\frac{a(x)}{\beta} z - z + 1} \right| \le \frac{\frac{\varepsilon}{R}}{1-\varepsilon} < \frac{2\varepsilon}{R}.$$

Moreover, using $1/|\beta| < 1/\varepsilon$ and $1/(1-\varepsilon) < 2$, we have

$$\left| \frac{a(x)}{\beta^2\left(z - \frac{1}{1-\frac{a(x)}{\beta}}\right)^2 \left(1-\frac{a(x)}{\beta}\right)^2} \right| = \left| \frac{a(x)}{\beta^2\left(\frac{a(x)}{\beta} z - z + 1\right)^2} \right| < \frac{32(L+1)}{\varepsilon^3}.$$

$<$Case 3$>$ $|a(x)| < 8(L+1)/\varepsilon$, $|a(x)/\beta - 1| > \varepsilon/R$, and $|a(x)/u - 1| > \varepsilon/(2R)$.

From $1/|\beta| < 1/\varepsilon$, $1/u < 1/\varepsilon$, and condition (b) mentioned above, we have

$$\left| \frac{1}{1-\frac{a(x)}{u}} - \frac{1}{1-\frac{a(x)}{\beta}} \right| = \left| \frac{a(x)hi}{u\beta(1-\frac{a(x)}{u})(1-\frac{a(x)}{\beta})} \right| < \frac{\varepsilon}{2}.$$

Therefore, as $a(x)/u \neq 1$ and $|z - 1/(1-a(x)/u)| > \varepsilon$, we obtain

$$\frac{1}{\left| z - \frac{1}{1-\frac{a(x)}{\beta}} \right|} < \frac{2}{\varepsilon}.$$

Moreover, this implies that

$$\left| \frac{a(x)}{\beta^2\left(z - \frac{1}{1-\frac{a(x)}{\beta}}\right)^2 \left(1-\frac{a(x)}{\beta}\right)^2} \right| < \frac{32(L+1)R^2}{\varepsilon^7}.$$

$<$Case 4$>$ $|a(x)| < 8(L+1)/\varepsilon$, $|a(x)/\beta - 1| > \varepsilon/R$, and $|a(x)/u - 1| \le \varepsilon/(2R)$.

This case is void as shown below:

$$\frac{\varepsilon}{2R} < \left|\frac{a(x)}{\beta} - 1\right| - \left|\frac{a(x)}{u} - 1\right| \le \left| \frac{a(x)}{\beta} - \frac{a(x)}{u} \right| = \frac{|ha(x)|}{|\beta u|} < \frac{\varepsilon^3}{4R^2},$$

which leads to the contradiction $4 < 2R < \varepsilon^2 < 1/4$.

From the inequalities mentioned above, the four integrands on the right-hand side of the following equalities are bounded. Therefore, the Cauchy-Riemann equations for $w_\beta(z)$ hold:

$$\frac{\partial}{\partial t} w_\beta(z) = \int_I \frac{-1}{\left(z - \frac{1}{1-\frac{a(x)}{\beta}}\right)^2} dF(x), \tag{6}$$

$$\frac{\partial}{\partial s}w_{\beta}(z)=\int_I \frac{-i}{\left(z-\frac{1}{1-\frac{a(x)}{\beta}}\right)^2}dF(x)=i\frac{\partial}{\partial t}w_{\beta}(z),$$

$$\frac{\partial}{\partial u}w_{\beta}(z)=\int_I \frac{-a(x)}{\beta^2\left(\frac{a(x)}{\beta}z-z+1\right)^2}dF(x),$$

$$\frac{\partial}{\partial h}w_{\beta}(z)=\int_I \frac{-a(x)i}{\beta^2\left(\frac{a(x)}{\beta}z-z+1\right)^2}dF(x)=i\frac{\partial}{\partial u}w_{\beta}(z). \quad \square$$

Moreover, the integrand of $w_{\beta}(z)$ is bounded, which insures that $w_{\beta}(z)$ is continuous with respect to β and z, respectively. Thus, $w_{\beta}(z)$ is holomorphic with respect to $(z,\ \beta)$. Henceforth, in this Section, we assume that $\max(0,\ \xi)<u<E$ and $0<t<u/(u-\xi)$. It should be noted that $a(x)t-ut+u\ge \xi t-ut+u=(u-\xi)(u/(u-\xi)-t)>0$ for each $x\in I$.

Specially, when $F(x)$ is a finite step function, we can write $X=\{(a_j,\ p_j)\}_{j=1,2,\dots,n}$ with $\xi=a_1=\min_j a_j,\ p_1>0$, $E=\sum_{j=1}^n a_j p_j>0$, $H=\sum_{j=1}^n p_j/a_j$, $H_{\xi}=+\infty$, $w_u(t)=\sum_{j=1}^n (a_j-u)p_j/(a_j t-ut+u)$, and so on.

Lemma 3.2. $w_u(t)$ *is strictly decreasing with respect to* t.

Proof. According to Lemma 3.1, we have

$$\frac{\partial}{\partial t}w_u(t)=-\int_I\left(\frac{a(x)-u}{a(x)t-ut+u}\right)^2 dF(x)<0. \quad \square$$

Lemma 3.3. $\lim_{t\to 0^+} w_u(t)=E/u-1$.

Proof. Since $(a(x)-u)/(a(x)t-ut+u)$ is decreasing with respect to t, using Lebesgue (monotone convergence) theorem, we obtain

$$\lim_{t\to 0^+} w_u(t)=\lim_{t\to 0^+}\int_I \frac{a(x)-u}{a(x)t-ut+u}dF(x)=\int_I \frac{a(x)-u}{u}dF(x)$$
$$=\frac{E}{u}-1. \quad \square$$

Lemma 3.4. $\lim_{t\to(u/(u-\xi))^-} w_u(t)=(1-\xi/u)H_{\xi}(\xi+1/H_{\xi}-u)$.

Proof. Using the same principle as above, we obtain

$$\lim_{t\to(u/(u-\xi))^-} w_u(t)=\frac{u-\xi}{u}\int_I \frac{a(x)-u}{a(x)-\xi}dF(x)$$
$$=\frac{u-\xi}{u}(1-(u-\xi)H_{\xi})=(1-\xi/u)H_{\xi}(\xi+1/H_{\xi}-u).$$

If $\xi=0$ and $H=+\infty$, then $\lim_{t\to 1^-} w_u(t)=1-u\int 1/a(x)dF(x)$
$=-\infty$. $\square$

From the above lemmas, if $\max(0,\xi+1/H_{\xi})<u<E$, then $\lim_{t\to 0^+} w_u(t)>0$ and $\lim_{t\to(u/(u-\xi))^-} w_u(t)<0$. Thus, the equation $w_u(t)=0$ has the only solution $\tilde{t}_u\in(0,\ u/(u-\xi))$, and we refer to it as *pre-optimal proportion*. Note that, due to Lemma 3.1 and the inverse mapping theorem, $\tilde{t}_u$ is analytic with respect to u.

Lemma 3.5. *If* $\xi>0,\ \ \xi+1/H_{\xi}<1/H$.

Proof. If $H_{\xi}=+\infty$, the assertion $H<1/\xi$ is obvious. Since $w_u(t)$ is strictly decreasing and $1<u/(u-\xi)$, $w_u(1)>\lim_{t\to(u/(u-\xi))^-} w_u(t)$, that is,

$$\int_I \frac{a(x)-u}{a(x)-u+u}dF(x)=1-Hu>(1-\xi/u)H_{\xi}(\xi+1/H_{\xi}-u)$$

for each $\xi<u<E$. If $1/H\le \xi+1/H_{\xi}$, then selecting $u=\xi+1/H_{\xi}<E$ leads to the contradiction that

$$0\ge 1-Hu>(1-\xi/u)H_{\xi}(\xi+1/H_{\xi}-u)=0. \quad \square$$

Lemma 3.6. *If* $\xi\ge 0$, *then* $\tilde{t}_u$ *is strictly decreasing with respect to* $u\in(\xi+1/H_{\xi},\ E)$.

Proof. From Lemma 3.1, $\tilde{t}_u$ is analytic. Using

$$\frac{\partial}{\partial u}\left(\frac{a(x)-u}{a(x)\tilde{t}_u-u\tilde{t}_u+u}\right)=-\frac{a(x)+(a(x)-u)^2\frac{d\tilde{t}_u}{du}}{(a(x)\tilde{t}_u-u\tilde{t}_u+u)^2},$$

$dw_u(\tilde{t}_u)/du=0$, and $a(x)\ge\xi\ge 0$, we obtain

$$\frac{d\tilde{t}_u}{du}=\frac{-\int_I \frac{a(x)}{(a(x)\tilde{t}_u-u\tilde{t}_u+u)^2}dF(x)}{\int_I \frac{(a(x)-u)^2}{(a(x)\tilde{t}_u-u\tilde{t}_u+u)^2}dF(x)}<0. \quad \square$$

Lemma 3.7. $\lim_{u\to E^-}\tilde{t}_u=0$.

Proof. < Case 1 >. Assume that $E=+\infty$. From $\lim_{u\to+\infty} u/(u-\xi)=1$, for any $0<\varepsilon<1/3$, there exists N such that $1-\varepsilon<u/(u-\xi)<1+\varepsilon$ for each $u>N$. This implies that

$$\left|\frac{a(x)-u}{a(x)t-ut+u}\right|=\begin{cases} 0, & \text{if } a(x)=u,\\ \dfrac{1}{\left|t-\frac{u}{u-a(x)}\right|}\le\dfrac{1}{\left|t-\frac{u}{u-\xi}\right|}<\dfrac{1}{\varepsilon}, & \text{if } a(x)<u,\\ \dfrac{1}{\left|t+\frac{u}{a(x)-u}\right|}\le\dfrac{1}{t}\le\dfrac{1}{\varepsilon}, & \text{if } a(x)>u,\end{cases}$$

for $\varepsilon\le t\le 1-2\varepsilon,\quad u>N$. Therefore, by Lebesgue (dominated-convergence) theorem, we obtain

$$\lim_{u\to+\infty} w_u(t)=\int_I \frac{1}{t-1}dF(x)=\frac{-1}{1-t}<0.$$

In particular, $\lim_{u\to+\infty} w_u(\varepsilon)=-1/(1-\varepsilon)<0$. Therefore, there exists $M>0$ such that $w_u(\varepsilon)<-1/(2(1-\varepsilon))$ for

each $u>M$. On the basis of the fact that $w_u(t)$ is strictly decreasing with respect to t, we have $0<\tilde{t}_u<\varepsilon$ for each $u>M$. This implies that $\lim_{u\to+\infty}\tilde{t}_u=0$.

<Case 2> Assume that $E<+\infty$. By Lemma 3.1, the analytic function $w_E(t)$ is well defined with respect to $t\in(0, \; E/(E-\xi))$. Even when $u=E<+\infty$, Lemmas 3.2 and 3.3 are valid. So, $w_E(t)$ is strictly decreasing and $\lim_{t\to 0^+} w_E(t)=0$. Therefore, we have $w_E(t)<0$.

If $0<\varepsilon<E/(2(E-\xi))$ and $(E+\max(0,\xi))/2<u<E$, then due to $0<\varepsilon<u/(u-\xi)$, $w_u(\varepsilon)$ is well defined. By Lemma 3.1 we have $\lim_{u\to E^-} w_u(\varepsilon)=w_E(\varepsilon)<0$. Therefore, there exists $\delta>0$ such that $w_u(\varepsilon)<0$ for each $u\in(E-\delta,E)$. This implies that $0<\tilde{t}_u<\varepsilon$ and

$\lim_{u\to E^-}\tilde{t}_u=0$. □

Lemma 3.8. *If* $\xi>0$, $\tilde{t}_{1/H}=1$.

Proof. If $\xi>0$, then by Lemma 3.5 we have $\xi+1/H_\xi<1/H<E$ and $1<u/(u-\xi)$. Therefore, $w_u(t)$ and $\tilde{t}_u$ are analytic near $(u,t)=(1/H, \; 1)$. The conclusion follows from the equality

$$w_{1/H}(1)=\int_I \frac{a(x)-\frac{1}{H}}{a(x)-\frac{1}{H}+\frac{1}{H}}dF(x)=0. \quad \square$$

Lemma 3.9. *If* $\xi>0$, $\lim_{u\to(\xi+1/H_\xi)^+}\tilde{t}_u=1+\xi H_\xi$.

Proof. Due to Lemma 3.6, $\lim_{u\to(\xi+1/H_\xi)^+}\tilde{t}_u$ exists, and we denote it by γ. It is clear that $\gamma>1$. According to the inequality $\tilde{t}_u<u/(u-\xi)$, we have $\gamma\le 1+\xi H_\xi$.

Assume $H_\xi<+\infty$ and $\gamma<1+\xi H_\xi$, then, for any $0<\varepsilon<\min((1+\xi H_\xi-\gamma)/3,\gamma/2)$, there exists $\delta>0$ such that

$$\left|\tilde{t}_u-\gamma\right|<\varepsilon \quad \text{and} \quad \left|\frac{u}{u-\xi}-(1+\xi H_\xi)\right|<\varepsilon$$

for each $u\in(\xi+1/H_\xi,\xi+1/H_\xi+\delta)$. This implies that

$$\left|\frac{a(x)-u}{a(x)t-ut+u}\right|=\begin{cases} 0, & \text{if } a(x)=u,\\ \dfrac{1}{\frac{u}{u-a(x)}-\tilde{t}_u}<\dfrac{1}{\frac{u}{u-\xi}-\tilde{t}_u}<\dfrac{1}{\varepsilon}, & \text{if } a(x)<u,\\ \dfrac{1}{\tilde{t}_u+\frac{u}{a(x)-u}}<\dfrac{1}{\tilde{t}_u}<\dfrac{1}{\varepsilon}, & \text{if } a(x)>u.\end{cases}$$

By Lebesgue theorem, we obtain

$$0=\lim_{u\to(\xi+1/H_\xi)^+}\int_I \frac{1}{\tilde{t}_u-\frac{1}{1-\frac{a(x)}{u}}}dF(x)=\int_I \frac{1}{\gamma-\frac{1}{1-\frac{a(x)}{\xi+1/H_\xi}}}dF(x).$$

This is a contradiction because the term on the right is positive, which is deduced from the fact that the function

$$\int_I \frac{1}{t-\frac{1}{1-\frac{a(x)}{\xi+1/H_\xi}}}dF(x)$$

is strictly decreasing from $E/(\xi+1/H_\xi)-1>0$ to 0 with respect to $t\in(0, \; 1+\xi H_\xi)$.

Assume $H_\xi=+\infty$ and $\gamma<+\infty$. Then, we have $0<1/(a(x)\gamma-\xi\gamma+\xi)\le 1/(a(x)\tilde{t}_u-\xi\tilde{t}_u+\xi)\le 1/\xi$ for each $\xi<u<E$. Therefore, by Lebesgue theorem, we obtain

$$\begin{aligned}0&=\lim_{u\to\xi^+}\int_I \frac{a(x)-u}{a(x)\tilde{t}_u-u\tilde{t}_u+u}dF(x)\\ &=\lim_{u\to\xi^+}\frac{1}{\tilde{t}_u}(1-u\int_I \frac{1}{a(x)\tilde{t}_u-u\tilde{t}_u+u}dF(x))\\ &=\frac{1}{\gamma}(1-\xi\int_I \frac{1}{a(x)\gamma-\xi\gamma+\xi}dF(x))>\frac{1}{\gamma}(1-\xi\int_I \frac{1}{\xi}dF(x))=0,\end{aligned}$$

which is a contradiction. This implies that if $H_\xi=+\infty$, $\gamma=1+\xi H_\xi=+\infty$. □

Lemma 3.10. *If* $\xi<0$ *and* $\xi+1/H_\xi>0$, $\lim_{u\to(\xi+1/H_\xi)^+}\tilde{t}_u=1+\xi H_\xi$.

Proof. Due to $1/H_\xi>-\xi>0$, we have $H_\xi<+\infty$. It should be noted that there exists $\delta>0$ such that $\tilde{t}_u$ is strictly increasing or decreasing in the interval $u\in(\xi+1/H_\xi, \; \xi+1/H_\xi+\delta)$, which is demonstrated in the proof of Lemma 3.16. Therefore, $\lim_{u\to(\xi+1/H_\xi)^+}\tilde{t}_u$ exists and denoted by γ. By the inequality $\tilde{t}_u<u/(u-\xi)$, we have $\gamma\le 1+\xi H_\xi$. Assume $\gamma<1+\xi H_\xi$, then, as in the proof of Lemma 3.9, we have a contradiction. □

Lemma 3.11. *If* $\xi=0$ *and* $1/H>0$, $\lim_{u\to(1/H)^+}\tilde{t}_u=1$.

Proof. It should be noted that $H<+\infty$. Due to Lemma 3.6, $\lim_{u\to(1/H)^+}\tilde{t}_u$ exists, and is denoted by γ. According to the relation $\tilde{t}_u<u/(u-\xi)=1$, we have $\gamma\le 1$. Assume $\gamma<1$, then as in the proof of Lemma 3.9, the function $\int_I (a(x)-H)\;/(a(x)t-Ht+H)dF(x)$ is strictly decreasing from $HE-1>0$ to 0 in the interval $t\in(0, \; 1)$, which leads to a contradiction. □

Lemma 3.12. *If* $\xi<0$ *and* $\xi+1/H_\xi\le 0$, $\lim_{u\to0^+}\tilde{t}_u=0$.

Proof. On the basis of the definition, $0<\tilde{t}_u<u/(u-\xi)$ and $\max(0,\ \xi+1/H_\xi)=0<u<E$. Therefore,

$$0\le\lim_{u\to0^+}\tilde{t}_u\le\lim_{u\to0^+}u/(u-\xi)=0. \qquad\square$$

Lemma 3.13. *If* $\xi\le0$, $\tilde{t}_u<\int_{a(x)\neq0}dF(x)$. $\square$

Proof. As $\xi\le0$, we have $0<\tilde{t}_u<u/(u-\xi)\le1$ for each $u\in(\max(0,\ \xi+1/H_\xi),\ E)$, and $a(x)\tilde{t}_u-u\tilde{t}_u+u$

$$=\begin{cases} u>0, & \text{if } a(x)=u,\\ u-(u-a(x))\tilde{t}_u>u(a(x)-\xi)/(u-\xi)\ge0, & \text{if } a(x)<u,\\ (a(x)-u)\tilde{t}_u+u>u>0, & \text{if } a(x)>u.\end{cases}$$

In this case, the equation $w_u(\tilde{t}_u)=0$ is equivalent to

$$\int_{a(x)\neq0}\frac{u}{a(x)\tilde{t}_u-u\tilde{t}_u+u}dF(x)+\frac{1}{1-\tilde{t}_u}\int_{a(x)=0}dF(x)=1.$$

And so,

$$\frac{-1}{-\tilde{t}_u+1}\int_{a(x)=0}dF(x)+\frac{1}{\tilde{t}_u}\int_{a(x)\neq0}dF(x)$$
$$=\frac{u}{\tilde{t}_u}\int_{a(x)\neq0}\frac{1}{a(x)\tilde{t}_u-u\tilde{t}_u+u}dF(x)>0,$$

which leads to $\tilde{t}_u<\int_{a(x)\neq0}dF(x)$. $\square$

Lemma 3.14. *If* $\xi=0$ *and* $H=+\infty$, $\lim_{u\to0^+}\tilde{t}_u=\int_{a(x)>0}dF(x)$.

Proof. Due to Lemma 3.6, $\lim_{u\to0^+}\tilde{t}_u$ exists, and we denote it by γ. We can choose $\delta>0$ such that $\gamma/2<\tilde{t}_u<\gamma$ for each $u\in(0,\ \delta)$. It should be noted that the equation $w_u(\tilde{t}_u)=0$ is equivalent to

$$\frac{-1}{-\tilde{t}_u+1}\int_{a(x)=0}dF(x)+\int_{a(x)>0}\frac{1}{\tilde{t}_u-\frac{1}{1-\frac{a(x)}{u}}}dF(x)=0.$$

Assume that $\gamma<1$, we have

$$\left|\frac{1}{\tilde{t}_u-\frac{1}{1-\frac{a(x)}{u}}}\right|<\begin{cases}\dfrac{1}{1-\gamma}, & \text{if } 0<a(x)<u,\\ \dfrac{2}{\gamma}, & \text{if } a(x)>u,\end{cases}$$

for each $u\in(0,\ \delta)$. In this case, by Lebesgue theorem, we obtain

$$\frac{-1}{-\gamma+1}\int_{a(x)=0}dF(x)+\frac{1}{\gamma}\int_{a(x)>0}dF(x)=0,$$

which implies that $\gamma=\int_{a(x)>0}dF(x)$.

In the case in which $\gamma=1$, due to Lemma 3.13, we have $\gamma\le\int_{a(x)>0}dF(x)\le1$. Thus, $\gamma=\int_{a(x)>0}dF(x)$. $\square$

Lemma 3.15. *The function* $\tilde{t}_u/u$ *is strictly decreasing with respect to* $u\in(\max(0,\ \xi+1/H_\xi),\ E)$.

Proof. Using the equality

$$\int_I\frac{a(x)-u}{a(x)\tilde{t}_u-u\tilde{t}_u+u}dF(x)=\int_I\frac{(a(x)-u)^2\tilde{t}_u+a(x)u-u^2}{\left(a(x)\tilde{t}_u-u\tilde{t}_u+u\right)^2}dF(x)=0$$

and by the proof of Lemma 3.6, we obtain

$$\frac{d}{du}\left(\frac{\tilde{t}_u}{u}\right)=-\frac{u\int_I\frac{a(x)}{(a(x)\tilde{t}_u-u\tilde{t}_u+u)^2}dF(x)+\tilde{t}_u\int_I\frac{(a(x)-u)^2}{(a(x)\tilde{t}_u-u\tilde{t}_u+u)^2}dF(x)}{u^2\int_I\frac{(a(x)-u)^2}{(a(x)\tilde{t}_u-u\tilde{t}_u+u)^2}dF(x)}$$
$$=-\frac{\int_I\frac{1}{(a(x)\tilde{t}_u-u\tilde{t}_u+u)^2}dF(x)}{\int_I\frac{(a(x)-u)^2}{(a(x)\tilde{t}_u-u\tilde{t}_u+u)^2}dF(x)}<0. \qquad\square$$

We define the continuous function $\bar{t}_u$ in the interval $[0,\ +\infty)$ as follows:

$$\text{If } \xi>0, \text{ then } \bar{t}_u:=\begin{cases}1, & \text{if } 0\le u\le1/H,\\ \tilde{t}_u, & \text{if } 1/H<u<E,\\ 0, & \text{if } u\ge E.\end{cases} \qquad (7)$$

$$\text{If } \xi=0, \text{ then } \bar{t}_u:=\begin{cases}\int_{a(x)>0}dF(x), & \text{if } u=0.\\ 1, & \text{if } 0<u\le1/H,\\ \tilde{t}_u, & \text{if } 1/H<u<E,\\ 0, & \text{if } u\ge E.\end{cases} \qquad (8)$$

$$\text{If } \xi<0, \text{ then } \bar{t}_u:=\begin{cases}\dfrac{u}{u-\xi}, & \text{if } 0\le u\le\max(0,\xi+1/H_\xi),\\ \tilde{t}_u, & \text{if } \max(0,\xi+1/H_\xi)<u<E,\\ 0, & \text{if } u\ge E.\end{cases} \qquad (9)$$

Using Lemma 3.15 and the above extensions, we have that $\bar{t}_u/u$ is strictly decreasing in the interval $0<u<E$. We denote the value of $\lim_{u\to0^+}\bar{t}_u/u$ by η.

Lemma 3.16. *If* $\xi<0$, *then the value* $0<u_{\max}<E$ *exists, which satisfies the following properties:*

(a) $\bar{t}_u$ *is strictly increasing in the interval* $0<u<u_{\max}$.

(b) $\bar{t}_u$ *is strictly decreasing in the interval* $u_{\max}<u<E$.

Proof. $\bar{t}_u/u=1/(u-\xi)$ is strictly decreasing in the interval $0<u<\xi+1/H_\xi$, if $\xi+1/H_\xi>0$. As the function $y:=\bar{t}_u/u$ is strictly decreasing with respect to u, $\bar{t}_u$ can be considered to be a function with a variable $y\in(0,\ \eta)$.

If $u\in(\max(0,\ \xi+1/H_\xi),\ E)$, $\bar{t}_u=\tilde{t}_u$. From $w_u(\tilde{t}_u)=0$ we have

$$\int_I\frac{1}{a(x)y-\bar{t}_u+1}dF(x)=1.$$

Thus,

$$\int_I \frac{a(x)-\frac{d\bar{t}_u}{dy}}{\left(a(x)y-\bar{t}_u+1\right)^2}dF(x)=0.$$

This implies that

$$\frac{d\bar{t}_u}{dy}=\frac{\int_I \frac{a(x)}{(a(x)y-\bar{t}_u+1)^2}dF(x)}{\int_I \frac{1}{(a(x)y-\bar{t}_u+1)^2}dF(x)}.$$

Denoting $\int_I \frac{1}{(a(x)y-\bar{t}_u+1)^2}dF(x)$ by s, we obtain

$$\frac{d^2\bar{t}_u}{dy^2}=\frac{1}{s^2}\left(\begin{array}{c} -2s\int_I \frac{a(x)(a(x)-\frac{d\bar{t}_u}{dy})}{(a(x)y-\bar{t}_u+1)^3}dF(x) \\ +2\int_I \frac{a(x)}{(a(x)y-\bar{t}_u+1)^2}dF(x)\times\int_I \frac{a(x)-\frac{d\bar{t}_u}{dy}}{(a(x)y-\bar{t}_u+1)^3}dF(x)\end{array}\right)$$

$$=\frac{-2}{s^3}\left(\begin{array}{c} s^2\int_I \frac{a(x)^2}{(a(x)y-\bar{t}_u+1)^3}dF(x) \\ -2s\int_I \frac{a(x)}{(a(x)y-\bar{t}_u+1)^2}dF(x)\int_I \frac{a(x)}{(a(x)y-\bar{t}_u+1)^3}dF(x) \\ +\left(\int_I \frac{a(x)}{(a(x)y-\bar{t}_u+1)^2}dF(x)\right)^2\times\int_I \frac{1}{(a(x)y-\bar{t}_u+1)^3}dF(x)\end{array}\right).$$

As a quadratic function with respect to s, $-s_3/2\times d^2\bar{t}_u/dy^2$ has the determinant given by

$$\left(\int_I \frac{a(x)}{(a(x)y-\bar{t}_u+1)^2}dF(x)\right)^2\times$$
$$\left(\begin{array}{c}\left(\int_I \frac{a(x)}{(a(x)y-\bar{t}_u+1)^3}dF(x)\right)^2 \\ -\int_I \frac{a(x)^2}{(a(x)y-\bar{t}_u+1)^3}dF(x)\times\int_I \frac{1}{(a(x)y-\bar{t}_u+1)^3}dF(x)\end{array}\right).$$

Due to Hölder inequality with respect to the two functions $\frac{a(x)}{(a(x)y-\bar{t}_u+1)^{3/2}}$ and $\frac{1}{(a(x)y-\bar{t}_u+1)^{3/2}}$, this determinant is negative. Therefore, we have $\frac{d^2\bar{t}_u}{dy^2}<0$, which implies that $\frac{d\bar{t}_u}{dy}$ is strictly decreasing.

First, we assume that $\xi+1/H_\xi\le 0$. Assign $\alpha:=\lim_{y\to 0^+} d\bar{t}_u/dy$ and $\beta:=\lim_{y\to\eta^-} d\bar{t}_u/dy$.

If $\alpha\le 0$, then $d\bar{t}_u/dy<0$ for each $0<y<\eta$. This contradicts the fact that $\bar{t}_u=0$ if $y=0$ (Lemma 3.7), and $\bar{t}_u>0$ if $0<y<\eta$. Therefore, $\alpha>0$.

If $\beta\ge 0$, $d\bar{t}_u/dy>0$ for each $0<y<\eta$. This contradicts the fact that $\bar{t}_u=0$ if $y=\eta$ (Lemma 3.11), and $\bar{t}_u>0$ if $0<y<\eta$. Therefore, $\beta<0$.

The value $0<y_{\max}<\eta$ such that $d\bar{t}_u/dy|_{y=y_{\max}}=0$ can then be determined. The value $0<u_{\max}<E$ required is determined using $y_{\max}$.

Second, we assume that $\xi+1/H_\xi>0$. If $u\in(0,\ \xi+1/H_\xi)$, then $\bar{t}_u=u/(u-\xi)$ is strictly increasing. Moreover, we have $u/(u-\xi)|_{u=\xi+1/H_\xi}=1+\xi H_\xi=\lim_{u\to(\xi+1/H_\xi)^+}\tilde{t}_u$ (Lemma 3.10) and $y|_{u=\xi+1/H_\xi}=H_\xi$. Redefine β as $\lim_{y\to H_\xi^-} d\bar{t}_u/dy$. If $\beta<0$, then as above, we obtain the required value $\xi+1/H_\xi<u_{\max}<E$. If $\beta\ge 0$, $d\bar{t}_u/dy>0$ for each $0<y<H_\xi$. This implies that $d\bar{t}_u/du=d\bar{t}_u/dy\times dy/du<0$ since $dy/du<0$ for each $u\in(\xi+1/H_\xi,\ E)$ (Lemma 3.15). Thus, $\bar{t}_u$ is strictly decreasing. Therefore, the required value is $u_{\max}=\xi+1/H_\xi$. □

Pre-Growth Rate

In this Section we assume that $u\in(\max(0,\ \xi),\ E)$ and $0<\rho<\ t<\ u/(u-\xi)$ unless otherwise mentioned. Define $G_{u,\rho}(t)$ by the value

$$\exp\left(\int_\rho^t w_u(t)dt\right)=\exp\left(\int_I \log\frac{a(x)t-ut+u}{a(x)\rho-u\rho+u}dF(x)\right),\quad (10)$$

which can be verified using the following inequalities:

$$\frac{u/(u-\xi)-t}{u/(u-\xi)-\rho}<\left|\frac{a(x)t-ut+u}{a(x)\rho-u\rho+u}\right|<\frac{t}{\rho},$$

$$\left|\frac{a(x)-u}{a(x)s-us+u}\right|<\max\left(\frac{1}{\rho},\frac{1}{u/(u-\xi)-t}\right)$$

for each $x\in I$ and $s\in(\min(\rho,\ t),\ \max(\rho,\ t))$. As $w_u(t)$ is strictly decreasing with respect to t from the positive value $E/u-1>0$, to the value $(1-\xi/u)H_\xi(\xi+1/H_\xi-u)$ (Lemmas 3.2, 3.3, and 3.4), $\int_\rho^t w_u(t)dt$ is strictly decreasing with respect to ρ near 0^+. Therefore, the limit

$$\lim_{\rho\to 0^+}\exp\left(\int_\rho^t w_u(t)dt\right)=\exp\left(\lim_{\rho\to 0^+}\int_I \log\frac{a(x)t-ut+u}{a(x)\rho-u\rho+u}dF(x)\right)\quad (11)$$

$$=\exp\left(\int_I \log\left(a(x)t/u-t+1\right)dF(x)\right)$$

finitely exists or $+\infty$, which we denote by $\widetilde{G}_u(t)$ and refer to as *pre-growth rate*. The equality mentioned above is obtained using Lebesgue theorem because the integrand is monotone with respect to ρ in $\{\ x\ |\ a(x)>u\ \}$ or $\{\ x\ |\ a(x)<u\ \}$.

Lemma 4.1. $\widetilde{G}_u(t)<E/u$, if $0<t<\min(1,u/(u-\xi))$.

Proof. By Jensen's inequality we have

$$\int_I \log\left(a(x)t/u-t+1\right)dF(x)\le\log\int_I\left(a(x)t/u-t+1\right)dF(x)$$
$$=\log(Et/u-t+1)<\log(E/u).\quad □$$

Lemma 4.2.

$$\int_{a(x)<u} \left|\log\left(a(x)t/u-t+1\right)\right| dF(x) < +\infty.$$

Proof. In general, $a(x)t/u-t+1 \geq (u-\xi)(u/(u-\xi)-t)/u > 0$. If $a(x)<u$, $a(x)t/u-t+1 < 1$. Therefore, we obtain

$$\int_{a(x)<u} \left|\log\left(a(x)t/u-t+1\right)\right| dF(x) \leq \left|\log\left(\frac{u-\xi}{u}\left(\frac{u}{u-\xi}-t\right)\right)\right| < +\infty.$$

□

Lemma 4.3. *The following three statements are equivalent.*

(1) $\int_{a(x)>1} \log a(x) dF(x) < +\infty.$

(2) $\widetilde{G}_u(t) < +\infty$ *for each* u *and* t.

(3) $\widetilde{G}_{u_1}(t_1) < +\infty$ *for some* u_1 *and* t_1.

Proof. (1) $\Rightarrow$ (2). The function $\log\left(a(x)t/u-t+1\right)$ satisfies the following inequalities.

$$\int_{u<a(x)\leq 1} \left|\log a(x)\right| dF(x) \leq \left|\log u\right| < +\infty,$$

$$\left|\int_{a(x)>\max(1,\,u)} \log\left(a(x)t/u-t+1\right) dF(x) - \int_{a(x)>\max(1,\,u)} \log a(x) dF(x)\right|$$

$$= \left|\int_{a(x)>\max(1,\,u)} \left(\log\frac{t}{u} + \log\left(1+\frac{u(1-t)}{a(x)t}\right)\right) dF(x)\right|$$

$$\leq \left|\log\frac{t}{u}\right| + \left|\log t\right| < +\infty.$$

Based on Lemma 4.2, we obtain the result.

(3) $\Rightarrow$ (1). It should be noted that $u_1 \in (\max(0,\ \xi),\ E)$ and $t_1 \in (0,\ u_1/(u_1-\xi))$. The result can be obtained in a similar manner as above.

(2) $\Rightarrow$ (3). It is clear. □

If one of the above three statements is satisfied, we can write $\widetilde{G} < +\infty$.

Lemma 4.4. *If* $\widetilde{G} < +\infty$, $\lim_{t\to 0^+} \widetilde{G}_u(t) = 1$.

Proof. Since $\lim_{t\to 0^+} w_u(t) = E/u-1 > 0$ (Lemma 3.3), $\int_0^t w_u(t)dt$ is strictly increasing and bounded with respect to t near 0^+. Therefore, we obtain that $\lim_{t\to 0^+} \int_0^t w_u(t)dt = 0.$ □

Lemma 4.5. *If* $u \in (\max(0,\ \xi+1/H_\xi),\ E)$, $\max_{0<t<u/(u-\xi)} G_{u,\rho}(t) = G_{u,\rho}(\widetilde{t}_u)$.

Proof. It is clear from the facts that $0 < G_{u,\rho}(t) < +\infty$, $(1-E/u)H_\xi(\xi+1/H_\xi-u) < w_u(t) < E/u-1$, and

$$\partial G_{u,\rho}(t)/\partial t = \frac{\partial}{\partial t}\exp\left(\int_\rho^t w_u(t)dt\right)$$
$$= G_{u,\rho}(t)\left(\frac{\partial}{\partial t}\int_\rho^t w_u(t)dt\right) = G_{u,\rho}(t) w_u(t). \quad \square$$

Lemma 4.6. *If* $\widetilde{G} < +\infty$ *and* $u \in (\max(0,\ \xi+1/H_\xi),\ E)$, $\max_{0<t<u/(u-\xi)} \widetilde{G}_u(t) = \widetilde{G}_u(\widetilde{t}_u)$.

Proof. In a similar manner as that of the proof of Lemma 4.5, we have

$$\frac{\partial \widetilde{G}_u(t)}{\partial t} = \frac{\partial}{\partial t}\lim_{\rho\to 0^+}\exp\left(\int_\rho^t w_u(t)dt\right) = \lim_{\rho\to 0^+}\frac{\partial}{\partial t}\exp\left(\int_\rho^t w_u(t)dt\right)$$
$$= \lim_{\rho\to 0^+} G_{u,\rho}(t) w_u(t) = \widetilde{G}_u(t) w_u(t),$$

which implies the conclusion. □

Lemma 4.7. *Two functions* $G_{u,\rho}(t)$ *and* $\widetilde{G}_u(t)$ $(<+\infty)$ *are concave with respect to* t.

Proof. Using Lemmas 3.2, 4.5, and Hölder inequality, we have

$$\partial^2 G_{u,\rho}(t)/\partial t^2 = G_{u,\rho}(t)(w_u^2(t) + \partial w_u(t)/\partial t)$$
$$= G_{u,\rho}(t)\begin{pmatrix} \left(\int_I (a(x)-u)/(a(x)t-ut+u) dF(x)\right)^2 \\ -\int_I (a(x)-u)^2/(a(x)t-ut+u)^2 dF(x) \end{pmatrix} < 0.$$

□

Along the similar way of discussions, we also have $\partial^2 \widetilde{G}_u(t)/\partial t^2 < 0$.

Lemma 4.8. *If* $\xi \geq 0$ *and* $t > \rho$, $G_{u,\rho}(t)$ *is strictly decreasing with respect to* u.

Proof. From $a(x) \geq 0$, we obtain

$$\frac{\partial G_{u,\rho}(t)}{\partial u} = \frac{\partial}{\partial u}\exp\left(\int_\rho^t w_u(t)dt\right) = G_{u,\rho}(t) \times \frac{\partial}{\partial u}\int_\rho^t w_u(t)dt$$
$$= -G_{u,\rho}(t) \times \int_\rho^t \left(\int_I \frac{a(x)}{(a(x)t-ut+u)^2} dF(x)\right) dt < 0. \quad \square$$

Lemma 4.9. *If* $\xi \geq 0$ *and* $\widetilde{G} < +\infty$, $\widetilde{G}_u(t)$ *is strictly decreasing with respect to* u.

Proof. Using Lemma 4.8, we obtain the conclusion. □

Lemma 4.10. *If* $\widetilde{G} < +\infty$, $\lim_{t\to(u/(u-\xi))^-} \widetilde{G}_u(t)$ $= \exp(\int_I \log\left(a(x)-\xi\right) dF(x))\ \ /(u-\xi)$ *for each* u.

Proof. If $a(x) > u$, $a(x)t/u-t+1$ is strictly increasing with respect to t. Therefore, using Lebesgue theorem, we have

$$0 \leq \lim_{t\to(u/(u-\xi))^-} \int_{a(x)>u} \log(a(x)t/u-t+1) dF(x)$$
$$= \int_{a(x)>u} \log\frac{a(x)-\xi}{u-\xi} dF(x) < +\infty.$$

On the other hand, if $a(x) < u$, then $a(x)t/u-t+1$ is

strictly decreasing with respect to t. Hence, using Lebesgue theorem, we have

$$\lim_{t\to(u/(u-\xi))^-}\int_{a(x)<u}\log(a(x)t/u-t+1)dF(x)$$

$$=\int_{a(x)<u}\log\frac{a(x)-\xi}{u-\xi}dF(x)<0,$$

which implies that

$$\lim_{t\to(u/(u-\xi))^-}\widetilde{G}_u(t)$$

$$=\exp\left(\int_{a(x)>u}\log\frac{a(x)-\xi}{u-\xi}dF(x)+\int_{a(x)<u}\log\frac{a(x)-\xi}{u-\xi}dF(x)\right)$$

$$=\exp\left(\int_I\log\left(a(x)-\xi\right)dF(x)\right)/(u-\xi).\qquad\square$$

As an expansion of the definition of $\widetilde{G}_u(t)$, we define $\widetilde{G}_u((u/(u-\xi))^-)$ by $\exp(\int_I\log\left(a(x)-\xi\right)dF(x))\ /(u-\xi)$ for each $u\in(\max(0,\ \xi),\ E)$.

Lemma 4.11. $\widetilde{G}_u(\tilde{t}_u)>1$ *if* $u\in(\max(0\,,\ \xi+1/H_\xi)\,,\ E)$.

Proof. If $0<t<\tilde{t}_u$, $w_u(t)>0$. Hence, we have $\widetilde{G}_u(\tilde{t}_u)=\exp(\int_0^{\tilde{t}_u}w_u(t)dt)>e^0=1.$ $\square$

Lemma 4.12. $\widetilde{G}_u(\tilde{t}_u)$ $(<+\infty)$ *is strictly decreasing with respect to* $u\in(\max(0\,,\ \xi+1/H_\xi)\,,\ E)$.

Proof. If $|a(x)|\le 2|\xi|$, we have

$$\left|\frac{a(x)}{a(x)t-ut+u}\right|=\left|\frac{a(x)}{(a(x)-\xi)t+\xi t-ut+u}\right|\le\frac{2|\xi|}{(u-\xi)(u/(u-\xi)-t)}.$$

On the other hand, if $a(x)>2|\xi|$, we have

$$\left|\frac{a(x)}{a(x)t-ut+u}\right|=\left|\frac{a(x)}{(a(x)-\xi)t+\xi t-ut+u}\right|\le\left|\frac{a(x)}{(a(x)-\xi)t}\right|<\frac{2}{t}.$$

Thus, by the definition of $\widetilde{G}_u(t)$, we have

$$\frac{\partial\widetilde{G}_u(t)}{\partial u}=\widetilde{G}_u(t)\int_I\frac{\partial}{\partial u}\log(a(x)t/u-t+1)dF(x)$$

$$=-\frac{t}{u}\widetilde{G}_u(t)\times\left(\int_I\frac{a(x)}{a(x)t-ut+u}dF(x)\right).$$

The definition $w_u(\tilde{t}_u)=0$ leads to $\int_I a(x)/(a(x)\tilde{t}_u-u\tilde{t}_u+u)dF(x)=1$. Therefore,

$$\frac{\partial\widetilde{G}_u(\tilde{t}_u)}{\partial u}=-\frac{\tilde{t}_u}{u}\widetilde{G}_u(\tilde{t}_u)<0.\qquad\square$$

Lemma 4.13. *If* $\widetilde{G}<+\infty$, $\lim_{u\to E^-}\widetilde{G}_u(\tilde{t}_u)=1$.

Proof. From Lemmas 4.11 and 4.12, $\lim_{u\to E^-}\widetilde{G}_u(\tilde{t}_u)\ge 1$ exists. Assume that $\xi\ge 0$, then from Lemmas 3.6 and 3.7, $\tilde{t}_u/(1-\tilde{t}_u)$ is strictly decreasing near $u=E^-$, and $\lim_{u\to E^-}\tilde{t}_u=0$. Applying Lebesgue theorem to the equality

$$\widetilde{G}_u(\tilde{t}_u)-1+\tilde{t}_u=(1-\tilde{t}_u)\left(\exp\left(\int_I\log\left(\frac{a(x)\tilde{t}_u}{u(1-\tilde{t}_u)}+1\right)dF(x)\right)-1\right),$$

we have

$$\lim_{u\to E^-}\left(\widetilde{G}_u(\tilde{t}_u)-1+\tilde{t}_u\right)=\exp\left(\int_I\log\left(0+1\right)dF(x)\right)-1=0.$$

This implies that $\lim_{u\to E^-}\widetilde{G}_u(\tilde{t}_u)=1$.

In the case in which $\xi<0$, Lemma 3.16 can be used as a substitution of Lemma 3.6 near $u=E^-$, where $\tilde{t}_u$ is strictly decreasing to 0 (Lemma 3.7). In order to apply Lebesgue theorem, it is sufficient to divide the above integration into two parts { $x\mid a(x)\ge 0$ } and { $x\mid a(x)<0$ }. $\square$

Lemma 4.14. *If* $\widetilde{G}<+\infty$ *and* $u\in(\max(0\,,\ \xi+1/H_\xi)\,,$ $\widetilde{G}_u(\tilde{t}_u)\ =\exp\left(\int_u^E\tilde{t}_u/udu\right).$

Proof. Using $\partial\widetilde{G}_u(\tilde{t}_u)/\partial u=-\tilde{t}_u/u\times\widetilde{G}_u(\tilde{t}_u)$ (Lemmas 4.12 and 4.13), we can solve the differential equation. $\square$

Lemma 4.15 *If* $\xi=0$ *and* $H=+\infty$, $\lim_{u\to 0^+}\widetilde{G}_u(\tilde{t}_u)=+\infty$.

Proof. Lemma 4.12 ensures the existence of $\lim_{u\to 0^+}\widetilde{G}_u(\tilde{t}_u)$, which is finite or $+\infty$. If $a(x)>0$, $a(x)/u$ is strictly decreasing with respect to u, and $\lim_{u\to 0^+}a(x)/u=+\infty$. Using Lebesgue theorem and $\int_{a(x)>0}dF(x)>0$, we have

$$\lim_{u\to 0^+}\widetilde{G}_u(\tilde{t}_u)\ge\underline{\lim}_{u\to 0^+}\widetilde{G}_u(\tfrac{1}{2})$$

$$\ge\underline{\lim}_{u\to 0^+}\frac{1}{2}\exp\left(\int_{a(x)>0}\log\frac{a(x)+u}{2u}dF(x)\right)=+\infty.\qquad\square$$

Lemma 4.16. *If* $\xi=0$, $H<+\infty$, *and* $\widetilde{G}<+\infty$, $\lim_{u\to(1/H)^+}\widetilde{G}_u(\tilde{t}_u)\ =H\exp\left(\int_I\log a(x)dF(x)\right).$

Proof. By definition, $\widetilde{G}<+\infty$ implies that $\int_{a(x)>1}\log a(x)dF(x)<+\infty$. From Jensen's inequality, we have

$$+\infty>\log H=\log\int_I\frac{1}{a(x)}dF(x)\ge\int_I\log\frac{1}{a(x)}dF(x),$$

which implies that $\int_I\log a(x)dF(x)>-\infty$. Therefore, $\log a(x)$ is integrable.

It should be noted that $\lim_{u\to(1/H)^+}\tilde{t}_u=1$ (Lemma 3.11). Using the equalities $\lim_{u\to(1/H)^+}\tilde{t}_u/u=H$ and $\lim_{u\to(1/H)^+}(1-\tilde{t}_u)u/\tilde{t}_u=0,$ we can choose $0<\delta<\min(1/H,\quad E-1/H)$, such that

$H/2<\tilde{t}_u/u<3H/2$ and $(1-\tilde{t}_u)u/\tilde{t}_u<1/2$ for each $u\in(1/H,\ 1/H+\delta)$. Therefore, we have the following properties.

(1) If $a(x)\ge 1/2$, then

$$\left|\log\left(a(x)\tilde{t}_u/u-\tilde{t}_u+1\right)\right|=\left|\log\frac{\tilde{t}_u}{u}+\log\left(a(x)+\frac{(1-\tilde{t}_u)u}{\tilde{t}_u}\right)\right|$$
$$<\max(\left|\log\frac{H}{2}\right|,\left|\log\frac{3H}{2}\right|)+\log 2+\left|\log a(x)\right|.$$

(2) If $a(x)<1/2$,

$$\left|\log\left(a(x)\tilde{t}_u/u-\tilde{t}_u+1\right)\right|<\max(\left|\log\frac{H}{2}\right|,\left|\log\frac{3H}{2}\right|)+\left|\log a(x)\right|.$$

Using the above properties, we can apply Lebesgue theorem as follows:

$$\lim_{u\to(1/H)^+}\widetilde{G}_u(\tilde{t}_u)=\lim_{u\to(1/H)^+}\exp\left(\int_I\log\left(a(x)\tilde{t}_u/u-\tilde{t}_u+1\right)dF(x)\right)$$
$$=H\exp\left(\int_I\log a(x)dF(x)\right).$$
□

Lemma 4.17. *If* $\xi=0$, $H<+\infty$, *and* $\widetilde{G}<+\infty$, $\lim_{u\to(1/H)^-}\widetilde{G}_u(1^-)\ =H\exp\left(\int_I\log a(x)dF(x)\right)$.

Proof. In the case in which $\xi=0$, based on the definition which is mentioned beneath the proof of Lemma 4.10, we have $\widetilde{G}_u(1^-)=\exp\left(\int_I\log a(x)dF(x)\right)/u$. Thus, we obtain the conclusion. □

Lemma 4.18. *If* $\xi=0$, $H<+\infty$, *and* $\widetilde{G}<+\infty$, $\lim_{u\to0^+}\widetilde{G}_u(1^-)\ =+\infty$.

Proof. Due to Lemmas 4.11, 4.12, 4.13, and 4.16, we have $H\ \ \exp(\int_I\log a(x)dF(x))\ \ >1$. Therefore, by the definition of $\widetilde{G}_u(1^-)$ we obtain

$$\lim_{u\to0^+}\widetilde{G}_u(1^-)=\lim_{u\to0^+}\exp\left(\int_I\log a(x)dF(x)\right)/u\ge\lim_{u\to0^+}1/H/u=+\infty.$$
□

Lemma 4.19. *If* $\xi>0$ *and* $H_\xi=+\infty$, $\lim_{u\to\xi^+}\widetilde{G}_u(\tilde{t}_u)=+\infty$.

Proof. Lemma 4.12 ensures the existence of $\lim_{u\to\xi^+}\widetilde{G}_u(\tilde{t}_u)$, which is finite or $+\infty$. If $a(x)>\xi$, then $(a(x)-\xi)/(2(u-\xi))$ is strictly decreasing with respect to $u\in(\xi\ ,\ E)$. Using Lebesgue theorem, we have

$$\lim_{u\to\xi^+}\widetilde{G}_u(\tilde{t}_u)\ge\underline{\lim}_{u\to\xi^+}\widetilde{G}_u(\frac{u}{2(u-\xi)})$$
$$\ge\underline{\lim}_{u\to\xi^+}\frac{1}{2}\exp\left(\int_{a(x)>\xi}\log\left(\frac{a(x)-\xi}{2(u-\xi)}+\frac{1}{2}\right)dF(x)\right)=+\infty.$$
□

Lemma 4.20. *If* $\xi>0$, $H_\xi<+\infty$, *and* $\widetilde{G}<+\infty$, $\lim_{u\to(\xi+1/H_\xi)^+}\widetilde{G}_u(\tilde{t}_u)\ =H_\xi\exp(\int_I\log\left(a(x)-\xi\right)dF(x))$.

Proof. An argument similar to that in the proof of Lemma 4.16 ensures that $\log(a(x)-\xi)$ is integrable.

It should be noted that $\lim_{u\to(\xi+1/H_\xi)^+}\tilde{t}_u=1+\xi H_\xi$ (Lemma 3.9). From the fact that $\lim_{u\to(\xi+1/H_\xi)^+}\tilde{t}_u/u=H_\xi$ and $\lim_{u\to(\xi+1/H_\xi)^+}(\xi\tilde{t}_u-u\tilde{t}_u+u)/\tilde{t}_u=0$, we can choose $0<\delta<\min(\xi+1/H_\xi,\quad E-\xi-1/H_\xi)$, such that $H_\xi/2<\tilde{t}_u/u<3H_\xi/2$ and $(\xi\tilde{t}_u-u\tilde{t}_u+u)/\tilde{t}_u<1/2$ for each $u\in(\xi+1/H_\xi,\ \xi+1/H_\xi+\delta)$. Therefore, we have the following properties.

(1) If $a(x)\ge\xi+1/2$,

$$\left|\log\left(a(x)\tilde{t}_u/u-\tilde{t}_u+1\right)\right|$$
$$=\left|\log\frac{\tilde{t}_u}{u}+\log\left(a(x)-\xi+\frac{\xi\tilde{t}_u-u\tilde{t}_u+u}{\tilde{t}_u}\right)\right|$$
$$<\max(\left|\log\frac{H_\xi}{2}\right|,\left|\log\frac{3H_\xi}{2}\right|)+\log 2+\left|\log\left(a(x)-\xi\right)\right|.$$

(2) If $a(x)<\xi+1/2$,

$$\left|\log\left(a(x)\tilde{t}_u/u-\tilde{t}_u+1\right)\right|<\max(\left|\log\frac{H_\xi}{2}\right|,\left|\log\frac{3H_\xi}{2}\right|)+\left|\log\left(a(x)-\xi\right)\right|.$$

Using the above properties, we can apply Lebesgue theorem as follows.

$$\lim_{u\to(\xi+1/H_\xi)^+}\widetilde{G}_u(\tilde{t}_u)=\lim_{u\to(\xi+1/H_\xi)^+}\exp\left(\int_I\log\left(a(x)\tilde{t}_u/u-\tilde{t}_u+1\right)dF(x)\right)$$
$$=H_\xi\exp\left(\int_I\log(a(x)-\xi)dF(x)\right).$$
□

Lemma 4.21. *If* $\xi>0$, $\widetilde{G}_{1/H}(\tilde{t}_{1/H})=H\exp\left(\int_I\log a(x)dF(x)\right)$.

Proof. It should be noted that $0<\xi<1/H<E$ and $1<1/H/(1/H-\xi)$. From Lemma 3.8, we have $\tilde{t}_{1/H}=1$. Thus,

$$\widetilde{G}_{1/H}(\tilde{t}_{1/H})=\exp\left(\int_I\log\left(a(x)/(1/H)-1+1\right)dF(x)\right)$$
$$=H\exp\left(\int_I\log a(x)dF(x)\right).$$
□

Lemma 4.22. *If* $\xi>0$, $\lim_{u\to(1/H)^-}\widetilde{G}_u(1)$ $=H\exp\left(\int_I\log a(x)dF(x)\right)$.

Proof. From $0<\xi<u$, we have $1<u/(u-\xi)$. Thus, by Lemma 4.21 we obtain

$$\lim_{u\to(1/H)^-}\widetilde{G}_u(1)=\widetilde{G}_{1/H}(1)=H\exp\left(\int_I\log a(x)dF(x)\right).$$
□

If $\xi>0$ and $0<u\le\xi$, $a(x)t/u-t+1\ge1$ for each $t>0$.

Therefore, we can expand the definition of $\widetilde{G}_u(t)=\exp(\int_I \log\left(a(x)t/u-t+1\right)dF(x))$, which is greater than 1 and finite or $+\infty$, in the domain $0<u\le\xi$ and $t>0$.

Lemma 4.23. *If* $\xi>0$, $\lim_{u\to 0^+}\ \widetilde{G}_u(1)=+\infty$.

Proof. It should be noted that $\widetilde{G}_{1/H}(\tilde{t}_{1/H})$ $=H\exp\left(\int_I \log a(x)dF(x)\right)>1$ (Lemma 4.21). From the expansion of $\widetilde{G}_u(t)$ which is defined beneath the proof of Lemma 4.22, we have

$$\lim_{u\to 0^+}\widetilde{G}_u(1)=\lim_{u\to 0^+}\exp\left(\int_I \log a(x)dF(x)\right)/u\ge\lim_{u\to 0^+}1/H/u=+\infty.$$

□

Lemma 4.24. *If* $\xi<0$, $\xi+1/H_\xi>0$ *and* $\widetilde{G}<+\infty$, $\lim_{u\to(\xi+1/H_\xi)^+}\ \widetilde{G}_u(\tilde{t}_u)\ =H_\xi\exp\left(\int_I \log\left(a(x)-\xi\right)dF(x)\right)$.

Proof. It should be noted that $H_\xi<+\infty$ and $\lim_{u\to(\xi+1/H_\xi)^+}\tilde{t}_u=1+\xi H_\xi$ (Lemma 3.10). The proof is formally the same as that of Lemma 4.20. □

Lemma 4.25. *If* $\xi<0$, $\xi+1/H_\xi>0$, *and* $\widetilde{G}<+\infty$, $\lim_{u\to(\xi+1/H_\xi)^-}\ \widetilde{G}_u((u/(u-\xi))^-)=H_\xi\exp\left(\int_I \log\left(a(x)-\xi\right)dF(x)\right)$.

Proof. We obtain the conclusion using the definition which is mentioned beneath the proof of Lemma 4.10.

Lemma 4.26. *If* $\xi<0$, $\xi+1/H_\xi>0$, *and* $\widetilde{G}<+\infty$, $\lim_{u\to 0^+}\widetilde{G}_u(u/(u-\xi)^-)\ =\exp\left(\int_I \log\left(a(x)-\xi\right)dF(x)\right)\ /(-\xi)$.

Proof. We obtain the conclusion by applying the same process as in Lemma 4.25. □

Lemma 4.27. *If* $\xi<0$ *and* $\xi+1/H_\xi<0$, $\eta<-1/\xi$.

Proof. It should be noted that the definition $w_u(\tilde{t}_u)=0$ implies that

$$\int_I \frac{1}{a(x)\frac{\tilde{t}_u}{u}-\tilde{t}_u+1}dF(x)=1.$$

From Lemma 3.12, we have $\lim_{u\to 0^+}\tilde{t}_u=0$. Using Fatou's lemma, we obtain

$$\int_I \frac{1}{a(x)\eta+1}dF(x)=\int_I \underline{\lim}_{u\to 0^+}\frac{1}{a(x)\frac{\tilde{t}_u}{u}-\tilde{t}_u+1}dF(x)\le 1.$$

Observe that $0<\eta=\lim_{u\to 0^+}\tilde{t}_u/u$ $\le\lim_{u\to 0^+}(u/(u-\xi))/u=-1/\xi$. Assume that $\eta=-1/\xi$, we have

$$\int_I \frac{1}{-a(x)/\xi+1}dF(x)=-\xi H_\xi\le 1.$$

This implies that $\xi+1/H_\xi\ge 0$, which is a contradiction. □

Lemma 4.28. *If* $\xi<0$ *and* $\xi+1/H_\xi=0$, $\eta=-1/\xi$.

Proof. Since $H_\xi<+\infty$, $1/(a(x)-\xi)$ is integrable. Thus, from $\int_I 1/(a(x)\tilde{t}_u/u-\tilde{t}_u+1)dF(x)=1$ and by Lebesgue theorem, we have $\int_I 1/(a(x)\eta+1)dF(x)=1$. On the other hand, it is clear that $\int_I 1/(a(x)\times 0+1)dF(x)=1$ and $\int_I 1/(a(x)(-1/\xi)+1)dF(x)=-\xi H_\xi=1$. This implies that the equation $\int_I 1/(a(x)y+1)dF(x)=1$ with respect to $y\in[0,\ -1/\xi]$ has three solutions $y=0,\ y=\eta,$ and $y=-1/\xi$. Note that

$$\frac{\partial^2}{\partial y^2}\int_I \frac{1}{a(x)y+1}dF(x)=\int_I \frac{a(x)^2}{\left(a(x)y+1\right)^3}dF(x)>0.$$

Therefore, the equation $\int_I 1/(a(x)y+1)dF(x)=1$ has at most two solutions. This implies that $\eta=-1/\xi$. □

Lemma 4.29. *If* $\xi<0$, $\xi+1/H_\xi\le 0$, *and* $\widetilde{G}<+\infty$, $\lim_{u\to 0^+}\widetilde{G}_u(\tilde{t}_u)\ =\exp\left(\int_I \log(a(x)\eta+1)dF(x)\right)$.

Proof. Lemma 3.12 implies that $\lim_{u\to 0^+}\tilde{t}_u=0$. From Lemma 3.15, $\tilde{t}_u/u$ is strictly decreasing with respect to $u\in(0\ ,\ E)$. Due to Lemma 3.16, $\tilde{t}_u$ is strictly increasing with respect to $u\in(0\ ,\ u_{\max})$. Therefore, if $a(x)>0$ then $a(x)\frac{\tilde{t}_u}{u}-\tilde{t}_u+1$ is strictly decreasing with respect to $u\in(0\,,\ u_{\max})$. This ensures that

$$\lim_{u\to 0^+}\int_{a(x)>0}\log(\frac{a(x)}{u}\tilde{t}_u-\tilde{t}_u+1)dF(x)=\int_{a(x)>0}\log(a(x)\eta+1)dF(x).$$

If $H_\xi<+\infty$, using Jensen's inequality, we see that $\log(a(x)-\xi)$ is integrable. If $a(x)\le 0$ and $0<u<\min(-\xi\ ,\ E)$, then we have

$$0\le\frac{a(x)-\xi}{-2\xi}\le\frac{a(x)-\xi}{u-\xi}<a(x)\frac{\tilde{t}_u}{u}-\tilde{t}_u+1<1,$$

and

$$\left|\log\left(a(x)\tilde{t}_u/u-\tilde{t}_u+1\right)\right|<\left|\log\left(a(x)-\xi\right)\right|+\left|\log(-2\xi)\right|.$$

Therefore, we can apply Lebesgue theorem to the following equality.

$$\lim_{u\to 0^+}\int_{a(x)\le 0}\log(\frac{a(x)}{u}\tilde{t}_u-\tilde{t}_u+1)dF(x)=\int_{a(x)\le 0}\log(a(x)\eta+1)dF(x).$$

Thus, we accomplish

$$\lim_{u\to 0^+}\widetilde{G}_u(\tilde{t}_u)=\exp\left(\int_I \log(a(x)\eta+1)dF(x)\right).$$

If $H_\xi=+\infty$, from Lemma 4.27, $\eta<-1/\xi$. If we assign $\varepsilon:=(\xi\eta+1)\ /2>0$, then, there exists $\delta>0$ such that $\tilde{t}_u<\varepsilon$ for each $u\in(0,\ \delta)$. Hence, if $a(x)\le 0$, then we have

$$1>\frac{a(x)}{u}\tilde{t}_u-\tilde{t}_u+1\ge\xi\eta-\varepsilon+1=\varepsilon.$$

Thus, we can apply Lebesgue theorem in the domain $\{\ x\mid a(x)<0\ \}$ and obtain the conclusion. □

Summing up the above-mentioned Lemmas, we obtain the following

Theorem 4.1. *If $\widetilde{G}<+\infty$, $\widetilde{G}_u(\bar{t}_u)$ is continuous and strictly decreasing with respect to $u\in(0,\ E)$. The range of $\widetilde{G}_u(\bar{t}_u)$ is* (1, $+\infty$) (*if* $\xi\ge 0$) *or* (1, $\exp\left(\int_I\log\left(a(x)\eta+1\right)dF(x)\right)$) (*if* $\xi<0$).

Double Sequence of Random Variables

It should be noted that a series of step functions exists such that $\xi\le f_N(x)\le f_{N+1}(x)\le a(x)$ and $\lim_{N\to+\infty}f_N(x)=a(x)$ for each $x\in I$, in which $\xi=\inf_{x\in I}a(x)>-\infty$ is the essential infimum.

For example, for each positive integer N, assign $M:=2^N N+1$ and

$$f_N(x):=\begin{cases}a_j:=\xi+\dfrac{j-1}{2^N}, & \text{if } \xi+\dfrac{j-1}{2^N}\le a(x)<\xi+\dfrac{j}{2^N}\ (1\le j\le M-1),\\ a_M:=\xi+N, & \text{if } a(x)\ge\xi+N.\end{cases}\tag{12}$$

In general, suppose $\{f_N(x)\mid x\in I\}\subset\{a_j\mid j=1,\dots,M\}$, where $\xi=a_1<a_2<\cdots<a_M<+\infty$. Set

$p_j:=\int_{a_j\le a(x)<a_{j+1}}dF(x)\ \ (1\le j\le M-1)$ and

$p_M:=\int_{a(x)\ge a_M}dF(x)$, then we have $\sum_{j=1}^M p_j=1$.

Assume $0<u$ (price), $0<t$ (proportion of investment), and $0<\xi t/u-t+1$. Then $0<a_jt/u-t+1$ for each $1\le j\le M$. For the random payoff $\{(a_j,\ \ p_j)\mid j=1,2,\cdots,M\}$, the growth rate per attempt after n attempts is

$$\left(\prod_{j=1}^M\left(a_jt/u-t+1\right)^{m_j}\right)^{\frac{1}{n}},\tag{13}$$

where a_j occurs m_j times $(m_1+m_2+\cdots+m_M=n)$ with probability $p_1^{m_1}\cdots p_M^{m_M}$. Such event has $n!\ /(m_1!m_2!\cdots m_M!)$ permutation patterns. We denote $X_{N,n}$ by this random variable. Then, the expectation $E[X_{N,n}]$ is expressed as

$$\sum_{m_1+m_2+\cdots+m_M=n}\frac{n!}{m_1!m_2!\cdots m_M!}\left(\prod_{j=1}^M\left(a_jt/u-t+1\right)^{m_j}\right)^{\frac{1}{n}}p_1^{m_1}\cdots p_M^{m_M}\tag{14}$$

$$=\left(\sum_{j=1}^M\left(a_jt/u-t+1\right)^{\frac{1}{n}}p_j\right)^n=\exp\left(\frac{\log\left(\sum_{j=1}^M\left(a_jt/u-t+1\right)^{\frac{1}{n}}p_j\right)}{\frac{1}{n}}\right)$$

$$=\exp\left(\frac{\log\left(\int_I\left(f_N(x)t/u-t+1\right)^{\frac{1}{n}}dF(x)\right)}{\frac{1}{n}}\right).$$

Moreover, the variance $V[X_{N,n}]$ is expressed as

$$\sum_{m_1+m_2+\cdots+m_M=n}\frac{n!}{m_1!m_2!\cdots m_M!}\left(\prod_{j=1}^M\left(a_jt/u-t+1\right)^{m_j}\right)^{\frac{2}{n}}p_1^{m_1}\cdots p_M^{m_M}-E[X_{N,n}]^2\tag{15}$$

$$=\exp\left(\frac{\log\left(\int_I\left(f_N(x)t/u-t+1\right)^{\frac{2}{n}}dF(x)\right)}{\frac{1}{n}}\right)-\exp\left(\frac{2\log\left(\int_I\left(f_N(x)t/u-t+1\right)^{\frac{1}{n}}dF(x)\right)}{\frac{1}{n}}\right).$$

Lemma 5.1.

$$\lim_{N\to+\infty}\left(\lim_{n\to+\infty}E[X_{N,n}]\right)=\exp\left(\int_I\log\left(a(x)t/u-t+1\right)dF(x)\right).$$

Proof. If n approaches $+\infty$, using l'Hôpital's theorem, we obtain

$$\lim_{n\to+\infty}E[X_{N,n}]=\lim_{h\to 0^+}\exp\left(\frac{\sum_{j=1}^M\left(a_jt/u-t+1\right)^h\log\left(a_jt/u-t+1\right)p_j}{\sum_{j=1}^M\left(a_jt/u-t+1\right)^h p_j}\right)$$

$$=\exp\left(\int_I\log\left(f_N(x)t/u-t+1\right)dF(x)\right).$$

Since $f_N(x)\le f_{N+1}(x)\le a(x)$, and $\lim_{N\to+\infty}f_N(x)=a(x)$, we obtain the conclusion. □

It is easily verified that if $\widetilde{G}<+\infty$, $\lim_{n\to+\infty}V[X_{N,n}]=0$ for each N.

Lemma 5.2. *If $v>0$, the following statements are equivalent.*

(1) $\int_{a(x)>1}a(x)^v\,dF(x)<+\infty$.

(2) $\int_I(a(x)t/u-t+1)^v\,dF(x)<+\infty$ *for each* $u>0$, $t>0$, *and* $\xi t/u-t+1>0$.

(3) $\int_I (a(x)t_1/u_1 - t_1 + 1)^{\nu} dF(x) < +\infty$ *for some* $u_1 > 0$, $t_1 > 0$, *and* $\xi t_1/u_1 - t_1 + 1 > 0$.

Proof. If $a(x) > 2u/t \times |1-t|$, we have $a(x) < 2u/t \times (a(x)t/u - t + 1)$ and $a(x)t/u - t + 1 < 2t/u \times a(x)$. This implies the conclusion. □

We say that a random payoff ($a(x)$, $F(x)$) is *effective* when $\int_{a(x)>1} a(x)^{\nu} dF(x) < +\infty$ for some $\nu > 0$, with the additional conditions $E > 0$ and $\xi > -\infty$.

It should noted that for each $0 < \nu < 1$, there exists h_{ν} such that $\log(x+1) < h_{\nu} x^{\nu}$ for each $x > 0$.

Lemma 5.3. *If a random payoff is effective,* $\widetilde{G} < +\infty$.

Proof. If $a(x) > 1$ and $0 < \nu < 1$, we have $\log a(x) < h_{\nu}(a(x)-1)^{\nu} < h_{\nu} a(x)^{\nu}$. If $a(x) > 1$ and $\nu \geq 1$, we have $\log a(x) < a(x) \leq a(x)^{\nu}$. Thus, we obtain the conclusion. □

We have some properties of the double sequence $E[X_{N,n}]$ as follows.

Lemma 5.4. *If a random payoff is effective,* $\lim_{N\to+\infty}\left(\lim_{n\to+\infty} E[X_{N,n}]\right) = \lim_{n\to+\infty}\left(\lim_{N\to+\infty} E[X_{N,n}]\right) = \lim_{\substack{n\to+\infty\\N\to+\infty}} E[X_{N,n}] < +\infty.$

If a random payoff is ineffective, $\lim_{N\to+\infty} E[X_{N,n}] = +\infty$ *for each* n, *if* $t > 0$.

Proof. For the assumption $\int_{a(x)>1} a(x)^{\nu} dF(x) < +\infty$, we can assume that $\nu < 1$. Suppose $0 < h < \nu/2$ and $a(x) > u$, we have

$$\left|\frac{\partial}{\partial h}(a(x)t/u - t + 1)^h\right| < (a(x)t/u - t + 1)^{\nu/2} \log(a(x)t/u - t + 1)$$
$$< h_{\nu/2}(a(x)t/u - t + 1)^{\nu}.$$

This guarantees that

$$\frac{\partial}{\partial h}\int_I (a(x)t/u - t + 1)^h dF(x) = \int_I \frac{\partial}{\partial h}(a(x)t/u - t + 1)^h dF(x) < +\infty.$$

Thus, since $f_N(x) \leq f_{N+1}(x) \leq a(x)$, and $\lim_{N\to+\infty} f_N(x) = a(x)$, we have

$$\lim_{n\to+\infty}\left(\lim_{N\to+\infty} E[X_{N,n}]\right)$$
$$= \exp\left(\lim_{h\to 0^+} \frac{\int_I (a(x)t/u - t + 1)^h \log(a(x)t/u - t + 1) dF(x)}{\int_I (a(x)t/u - t + 1)^h dF(x)}\right)$$
$$= \exp\left(\int_I \log\left(a(x)t/u - t + 1\right) dF(x)\right) < +\infty.$$

Using Lemma 5.1, we have

$$\lim_{N\to+\infty}\left(\lim_{n\to+\infty} E[X_{N,n}]\right) = \lim_{n\to+\infty}\left(\lim_{N\to+\infty} E[X_{N,n}]\right).$$

Set $\alpha := \lim_{N\to+\infty}\left(\lim_{n\to+\infty} E[X_{N,n}]\right) = \lim_{n\to+\infty}\left(\lim_{N\to+\infty} E[X_{N,n}]\right)$, $U_N := \lim_{n\to+\infty} E[X_{N,n}]$, and $W_n := \lim_{N\to+\infty} E[X_{N,n}]$ Since $f_N(x) \leq f_{N+1}(x) \leq a(x)$, $E[X_{N,n}]$ increases with respect to N. For any $\varepsilon > 0$, there exits N_0 and n_1 such that $|U_N - \alpha| < \varepsilon$ and $|W_n - \alpha| < \varepsilon$ for each $N \geq N_0$ and $n \geq n_1$. By setting $h := 1/n$ and applying l'Hôpital's theorem twice, we have

$$\lim_{h\to 0^+} \frac{\partial}{\partial h} E[X_{N,n}]$$
$$= \frac{\exp\left(\int_I \log(f_N(x)t/u - t + 1) dF(x)\right)}{2} \times$$
$$\left(\begin{array}{l}\int_I \left(\log(f_N(x)t/u - t + 1)\right)^2 dF(x) \\ -\left(\int_I \log(f_N(x)t/u - t + 1) dF(x)\right)^2\end{array}\right) > 0, \text{ if } p_1 < 1.$$

This implies that $E[X_{N,n}]$ decreases with sufficiently large n ($\partial/\partial n = -h^2 \partial/\partial h$). Then, there exits $n_2 = n_2(N_0) \geq n_1$ such that

$$E[X_{N_0,n_2}] > E[X_{N_0,n}] > E[X_{N_0,n+1}] \geq U_{N_0}$$

for each $n \geq n_2$. Therefore, we have

$$\alpha - \varepsilon < U_{N_0} < E[X_{N_0,n}] \leq E[X_{N,n}] \leq W_n < \alpha + \varepsilon,$$

which implies $\lim_{\substack{n\to+\infty\\N\to+\infty}} E[X_{N,n}] = \alpha$ as a double sequence.

If $\int_{a(x)>1} a(x)^{\nu} dF(x) = +\infty$ (ineffective) for each $\nu > 0$, using Lemma 5.2, we have

$$\lim_{N\to+\infty} E[X_{N,n}] = \exp\left(\frac{\log\left(\int_I (a(x)t/u - t + 1)^{\frac{1}{n}} dF(x)\right)}{\frac{1}{n}}\right) = +\infty$$

for each $n \geq 1$ and $t > 0$. □

Lemma 5.5. *If a random payoff is effective,* $\lim_{\substack{n\to+\infty\\N\to+\infty}} V[X_{N,n}] = 0.$

Proof. This can be proved in a manner similar to Lemma 5.4. □

Lemma 5.6. *If a random payoff is effective,*

$$\lim_{\substack{n\to+\infty\\N\to+\infty}} E[\left(X_{N,n} - \exp\left(\int_I \log\left(a(x)t/u - t + 1\right) dF(x)\right)\right)^2] = 0.$$

Proof. The equality $E[\left(X_{N,n} - c\right)^2] = V[X_{N,n}] + \left(E[X_{N,n}] - c\right)^2$ for each c implies the conclusion (Lemmas 5.1, 5.4, and 5.5). □

We denote $\exp(\int_I \log(a(x)t/u-t+1)dF(x))$ by $G_u(t)$ on the region $u>0$, $0\le t\le 1$, and $\xi t/u-t+1>0$. We refer to this as *the limit expectation of growth rate*. Notice that $\tilde{G}_u(t)$ is defined above Lemma 4.1 on the region $\max(0,\xi)<u<E$ and $0<t<u/(u-\xi)$, or $\xi>0$, $0<u\le\xi$, and $t>0$. It is clear that $\tilde{G}_u(t)=G_u(t)$ on the intersection of the regions.

It is said that t_u is *the optimal proportion of investment* in order to continue to invest without borrowing with respect to $u>0$, if

$$\overline{\lim}_{\substack{\rho\to t_u \\ 0\le\rho\le 1 \\ \xi\rho/u-\rho+1>0}} \int_I \log\frac{a(x)t/u-t+1}{a(x)\rho/u-\rho+1}dF(x)\le 0 \qquad (16)$$

for each $0\le t\le 1$ and $\xi t/u-t+1>0$. It follows that $0\le t_u\le 1$ and $\xi t_u/u-t_u+1\ \ge 0$. The existence and uniqueness of t_u for each $u>0$ is proved in Theorem 5.1.

If $\lim_{\rho\to t,\,0\le\rho\le 1,\,\xi\rho/u-\rho+1>0} G_u(t)$ exists, we extend $G_u(t)$ by the value if $\xi t/u-t+1\ =0$. Suppose $\tilde{G}<+\infty$, then, from $\log(G_u(t)/G_u(\rho)) = \int_I \log((a(x)t/u-t+1)/(a(x)\rho/u-\rho+1))\ dF(x)$, the equation $G_u(t_u) = \sup_{0\le t\le 1,\,\xi t/u-t+1>0} G_u(t)$ implies that t_u is the optimal proportion.

Theorem 5.1. $t_u=\bar{t}_u\ (u>0)$, *where* $\bar{t}_u$ *is a continuous function defined by* (7), (8) *and* (9) *beneath Lemma 3.15.*

Proof.

(1) If $u\ge E$, $t_u=0$.

It should be noted that Lemmas 3.2, 3.3 and equality (10) are valid even if $u\ge E$. Using $\partial w_u(t)/\partial t<0$ and $w_u(0^+)=E/u-1\le 0$, we have $w_u(t)<0$ for each $0<t<\min(1,\ u/(u-\xi))$. Therefore, we have

$$\int_\rho^t w_u(t)dt=\int_I \log\frac{a(x)t-ut+u}{a(x)\rho-u\rho+u}dF(x)<0$$

for each $0<\rho<t<\min(1,\ u/(u-\xi))$, and $\int_\rho^t w_u(t)>0$ for each $0<t<\rho\ <\min(1,\ u/(u-\xi))$. This implies that $t_u=0$..

(2) If $\xi\le 0$ and $0<u\le\xi+1/H_\xi$, $t_u=u/(u-\xi)$.

Notice that the condition $\xi=0$, $H=0$, and $0<u\le\xi+1/H_\xi$ is void. Using $w_u(u/(u-\xi)^-) =(1-\xi/u)H_\xi(\xi+1/H_\xi-u)\ge 0$ (Lemma 3.4), we have $w_u(t)>0$ (Lemma 3.2) for each $0<t<u/(u-\xi)$. The facts that $\int_\rho^t w_u(t)dt>0$ for each $0<\rho<t<u/(u-\xi)$, and $\int_\rho^t w_u(t)dt<0$ for each $0<t\ <\rho<u/(u-\xi)$ imply that $t_u=u/(u-\xi)$.

(3) If $\xi>0$ and $\xi<u\le\xi+1/H_\xi$, $t_u=1$.

Since $u/(u-\xi)>1$, we can show that $w_u(t)>0$ for each $0<t<1$ as shown in (2).

(4) If $\xi>0$ and $\xi+1/H_\xi<u\le 1/H$, $t_u=1$.

From Lemmas 3.6 and 3.8, we have $\tilde{t}_u\ge 1=\tilde{t}_{1/H}$ for each $u\in(\xi+1/H_\xi,\ 1/H]$. Therefore, from $w_u(\tilde{t}_u)=0$ and Lemma 3.2, we have $w_u(t)>0$ for each $0<t<1\le\tilde{t}_u$. Thus, we obtain the conclusion as shown in (3).

(5) If $\xi>0$ and $0<u\le\xi$, $t_u=1$.

From $u\le\xi\le a(x)$, we have $1\le a(x)\rho/u-\rho+1 \le a(x)t/u-t+1$ for each $0<\rho<t<1$. Therefore, $\int_I \log((a(x)t/u-t+1)/(a(x)\rho/u-\rho+1))dF(x)>0$ for each $0<\rho<t<1$ and $\int_I \log((a(x)t/u-t+1)/(a(x)\rho/u-\rho+1))\ dF(x)<0$ for each $0<t<\rho<1$. This implies $t_u=1$.

(6) If $\xi>0$ and $1/H<u<E$, $t_u=\tilde{t}_u$.

It should be noted that $u/(u-\xi)>1$. It is sufficient to show that $\int_{\tilde{t}_u}^t w_u(t)dt\ <0$ for each $0<t\ne\tilde{t}_u<1$. From Lemmas 3.6, 3.7 and 3.8, we have $0<\tilde{t}_u<1$. Moreover, from $w_u(\tilde{t}_u)=0$, we have $w_u(t)>0$ for each $0<t<\tilde{t}_u$ and $w_u(t)<0$ for each $\tilde{t}_u<t<1$. Therefore, we obtain the conclusion.

(7) If $\xi\le 0$ and $\max(0,\ \xi+1/H_\xi)<u<E$, $t_u=\tilde{t}_u$.

It should be noted that $u/(u-\xi)\le 1$. It is sufficient to show that $\int_{\tilde{t}_u}^t w_u(t)dt\ <0$ for each $0<t\ne\tilde{t}_u<u/(u-\xi)$. From Lemmas 3.2, 3.3 and 3.4, we have $0<\tilde{t}_u <u/(u-\xi)$. Thus, we obtain the conclusion as shown in (6). □

Hereafter, we assume that $\tilde{G}<+\infty$. Thus, it is easy to verify the following corollaries.

Corollary 5.1. *Suppose* $\xi\ge 0$ *and* $1/H<u<E$, *or* $\xi<0$ *and* $\max(0,\ \xi+1/H_\xi)<u<E$. *Then, the optimal proportion of investment* t_u *is uniquely determined by* $\int_I (a(x)-u)/(a(x)t_u-ut_u+u)dF(x)=0$, *and the maximized limit expectation of the growth rate is* $\exp\left(\int_I \log(a(x)t_u/u-t_u+1)dF(x)\right)$.

Corollary 5.2. *Suppose* $\xi < 0$ *and* $0 < u \le \xi + 1/H_\xi$. *Then, the optimal proportion of investment is* $u/(u-\xi)$, *and the maximized limit expectation of the growth rate is* $\exp(\int_I \log(a(x)-\xi)dF(x))/(u-\xi)$.

Corollary 5.3. *Suppose* $\xi \ge 0$ *and* $0 < u \le 1/H$. *Then, the optimal proportion of investment is* 1, *and the maximized limit expectation of the growth rate is* $\exp(\int_I \log a(x)\, dF(x))/u$.

Corollary 5.4. *Suppose* $u \ge E$. *Then, the optimal proportion of investment is* 0, *and the maximized limit expectation of the growth rate is* 1.

Geometic Pricing

In order to determine the price u^X of the effective random payoff $X = (a(x), F(x))$, we require the risk-free (simple or continuously compounded) interest rate $r > 0$ for a period. As $G_u(t_u)$ is strictly decreasing (Theorems 4.1 and 5.1), the solution of the equation $G_u(t_u) = r+1$ (if r is simple) or $G_u(t_u) = e^r$ (if r is continuously compounded) is uniquely determined. The latter case $G_u(t_u) = e^r$ is rewritten as:

Theorem 6.1 (the geometric price of a random payoff). *Consider an effective random payoff* $X = (a(x), F(x))$. *The geometric price* u^X *and optimal proportion of investment,* t_{u^X}, *with respect to a continuously compounded risk-free rate* $r > 0$, *are uniquely given by the equation*

$$\sup_{\substack{0\le t\le 1\\ t\le \xi t/u+1}} \int_I \log\left(a(x)\,t/u - t + 1\right)dF(x) = r,$$

except for the following two cases, where no solution exists.

(1) $\xi < 0$, $\xi + 1/H_\xi \le 0$ *and* $r \ge \int_I \log(a(x)\eta+1)dF(x)$, *where* η *is calculated by the equation* $\int_I 1/(a(x)\eta+1)dF(x) = 1$.

(2) $\xi < 0$, $\xi + 1/H_\xi > 0$ *and*

$$r \ge \int_I \log(a(x)-\xi)dF(x)) - \log(-\xi).$$

To calculate the values u^X *and* t_{u^X}, we have the following three special cases.

(3) $t_{u^X} = 1$ and $u^X = \exp(\int_I \log a(x)dF(x) - r)$ if $\xi > 0$ and $r \ge \log H + \int_I \log a(x)dF(x)$.

(4) $t_{u^X} = 1$ and $u^X = \exp(\int_I \log a(x)dF(x) - r)$ if $\xi = 0$, $H < +\infty$ and $r \ge \log H + \int_I \log a(x)dF(x)$.

(5) $t_{u^X} = u^X/(u^X - \xi)$ and $u^X = \xi + \exp(\int_I \log(a(x)-\xi)dF(x) - r)$ if $\xi < 0$, $\xi + 1/H_\xi > 0$, and $\log H_\xi + \int_I \log(a(x)-\xi)dF(x) \le r < \int_I \log(a(x)-\xi)dF(x) - \log(-\xi)$.

Apart from the five cases listed above, u^X *and* t_{u^X} are uniquely determined by the simultaneous equations

$$\begin{cases} \int_I u/(a(x)t - ut + u)dF(x) = 1, \\ \int_I \log\left(a(x)\,t/u - t + 1\right)dF(x) = r. \end{cases}$$

In this section we assume that $r = 0.04$. It is easy to verify that the following examples are effective.

Example 6.1. Suppose that the payoff and distribution functions are given by $a(x) = x$ and $F(x) = x \in I = [0, 1]$ respectively, then $\xi = 0$, $E = 1/2$, and $H = +\infty$. Set $y = t_u/u$ $(0 < u < 1/2)$, then the equation $w_u(t_u) = 0$ can be reduced to $\int_I 1/(xy - t_u + 1)dx = 1$. This integral equation has the solution $t_u = (e^y - y - 1)/(e^y - 1)$. Therefore, we obtain

$$G_u(t_u) = \frac{y}{e^y - 1}\exp\left(y - 1 + \frac{y}{e^y - 1}\right),$$

which strictly increases from 1 to $+\infty$ with respect to $y \in (0, +\infty)$. The price u should be the solution of the equation $G_u(t_u) = e^{0.04}$. Thus, we have $y \fallingdotseq 0.9918$, $u \fallingdotseq 0.4187$ and $t_u \fallingdotseq 0.4152$.

In the prevailing pricing theory, $E/u = e^{0.04}$ produce $u \fallingdotseq 0.4804$. In this case, from $\int_0^1 u/(xt_u - ut_u + u)\, dx = 1$ we have $t_u \fallingdotseq 0.1131$, and $G_u(t_u) \fallingdotseq 1.0023$ $(< e^{0.04} \fallingdotseq 1.0408)$.

Example 6.2. Suppose that the payoff a or b $(a > 1 > b)$ occurs with probability p or $q = 1-p$, respectively. Further assume that $1/H < u = 1 < E$ (if $b > 0$) or $u = 1 < E$ (if $b \le 0$). Then, from $p/(at_1 - t_1 + 1) + q/(bt_1 - t_1 + 1) = 1$, we obtain

$$t_1 = \frac{q}{1-a} + \frac{p}{1-b}, \qquad (17)$$

$$G_1(t_1) = (a-b)\left(\frac{q}{a-1}\right)^q\left(\frac{p}{1-b}\right)^p.$$

Samuelson (1971) deals with the case in which $a = 2.7$, $b = 0.3$, and $p = q = 0.5$, where $\xi = 0.3$, $E = 1.5$, $1/H = 0.54$ and $t_1 = 50/119 \fallingdotseq 0.4202$. However, Samuelson (1971) may have misinterpreted the criterion to be the geometric mean $2.7^{0.5}0.3^{0.5} = 0.9 < 1$, instead of $G_1(t_1) = (2.7-0.3)(0.5/1.7)^{0.5}(0.5/0.7)^{0.5}$

$\doteqdot 1.1000 > 1$.

Example 6.3. Consider a two person matrix game with the payoff matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 21 & -3 \\ -9 & 12 \end{pmatrix}$ for the row player A. Under the mixed strategies $\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} xy & x(1-y) \\ (1-x)y & (1-x)(1-y) \end{pmatrix}$, the geometric price $u = u(x, \ y)$ and optimal proportion $t = t_{u(x,y)}$ of investment are uniquely determined by the simultaneous equations:

$$\begin{cases} \sum_{i,j} up_{ij} / (a_{ij}t - ut + u) = 1, \\ \sum_{i,j} p_{ij} \log\left(a_{ij} \ t / u - t + 1\right) = r, \end{cases} \tag{18}$$

if E= $\sum_{i,j} a_{ij} p_{ij} > 0$ and $(x, \ y) \notin \{(0, \ 0), \ (0, \ 1), \ (1, \ 0), \ (1, \ 1)\}$. For each $0 \le x \le 1$ there exits $0 \le y \le 1$ such that $E \le 0$ or $\sum_{i,j} p_{ij} \log\left(a_{ij} \ \eta + 1\right) < r = 0.04$. Thus, the row player A has no solution for investment (Theorem 6.1 (1)).

In the prevailing pricing theory, $E / u = e^r$ with the mixed strategies $x = 7/15$ and $y = 1/3$ produces that $u = 5 / e^{0.04} \doteqdot 4.8039$, $t_u \doteqdot 0.0084$ and $G_u(t_u) \doteqdot 1.0002$ ($< e^{0.04} \doteqdot 1.0408$). Under the fixed conditions $x = 7/15$, $y = 1/3$, and $r = 0.0002$, we have the geometric price $u \doteqdot 4.7885$ with $t_u \doteqdot 0.0091$.

Example 6.4. In the St. Petersburg game (Bernoulli 1954), suppose that the payoff 2^j occurs with probability $1/2^j$ ($j = 1, 2, \cdots$.), then $\xi = 2$, $E = +\infty$, and $H = 1/3$. This game is effective, because $\sum_{j=1}^{\infty} (2^j)^{1/2} / 2^j = 1/(\sqrt{2} - 1) < +\infty$. From Lemma 4.21 we have $G_{1/H}(t_{1/H}) = 1/3 \times \exp(\sum_{j=1}^{\infty} (\log 2^j)/2^j) = 4/3$. Thus, $G_u(t_u)$ ($u \in (3, +\infty)$) strictly decreases from $4/3$ to 1. The equation $G_u(t_u) = e^{0.04}$ yields the price $u \doteqdot 5.0815$. Therefore, if the investors invest $t_u \doteqdot 0.1686$ of their current capital, they can maximize the limit expectation of growth rate to $e^{0.04}$.

In the prevailing pricing theory, $u = E / e^{0.04} = +\infty / e^{0.04}$ yields the infinity price.

Example 6.5. The lognormal distributed random payoff is given by

$$a(x) = Se^r e^x, \quad dF(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x+\sigma^2/2)^2}{2\sigma^2}} dx, \tag{19}$$

and $I = (-\infty, +\infty)$. In this case, we have $E = Se^r$, $H = (e^{-r+\sigma^2}) / S$, and $\exp\left(\int_I \log a(x) dF(x)\right) = Se^{r - \sigma^2/2}$.

When $S = 100$, $\sigma = 0.3$, and $r = 0.04$, we have $\xi = 0$, $E \doteqdot 104.0811$, $H \doteqdot 0.0105127$, and $1/H \doteqdot 95.1229$. From Lemma 4.16, $G_u(t_u)$ ($u \in (1/H, \ E)$) strictly decreases from $H \exp\left(\int_I \log a(x) dF(x)\right) = e^{\sigma^2/2} \doteqdot 1.0460$ to 1. The equation $G_u(t_u) = e^{0.04} \doteqdot 1.0408$ yields the price $u \doteqdot 95.6132$. Therefore, if the investors invest $t_u \doteqdot 0.9450$ of their current capital, then they can maximize the limit expectation of growth rate to $e^{0.04}$.

In the prevailing pricing theory, $E / u = e^r$ yields the (higher) price $u = 100$ (> 95.6132). Under this price, if the investors invest $t_u \doteqdot 0.4433$ of their current capital, they can maximize the limit expectation of growth rate to 1.0088 ($< e^{0.04} \doteqdot 1.0408$). Because $\exp(r - \sigma^2/2) \doteqdot 0.9950$ (< 1.0088), the statement (Luenberger 1998) that the expected growth rate is equal to $r - \sigma^2/2$ is not necessarily true.

Example 6.6. The European put option is given by

$$a(x) = \max(K - Se^{rT}e^x, 0), \quad dF(x) = \frac{1}{\sqrt{2\pi T}\sigma} e^{-\frac{(x+\sigma^2 T/2)^2}{2\sigma^2 T}} dx, \tag{20}$$

and $I = (-\infty, +\infty)$. We assume that the stock price $Y = Se^{rT}e^X$ ($X = (x, \ F(x))$) is lognormally distributed with volatility $\sigma\sqrt{T}$, where S is the current stock price, r is the continuously compounded interest rate, K is the exercise price of the put option, and T is the exercise period. The expectation E of $(a(x), \ F(x))$ is given by

$$E = \frac{1}{\sqrt{2\pi T}\sigma} \int_{-\infty}^{\log\frac{K}{S} - rT} (K - Se^{rT}e^x) e^{-\frac{(x+\sigma^2 T/2)^2}{2\sigma^2 T}} dx \tag{21}$$

$$= KN\left(-\frac{\log\frac{S}{K} + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) - Se^{rT} N\left(-\frac{\log\frac{S}{K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right),$$

where $N(x) = \int_{-\infty}^{x} e^{-t^2/2} / \sqrt{2\pi} dt$ is the cumulative standard normal distribution function.

When $S = 90$, $K = 120$, $T = 2$, $\sigma = 0.1$, and $r = 0.04$, we have $\xi = 0$, $E \doteqdot 22.9848$, and $H = +\infty$. Therefore, from Theorems 4.1 and 5.1, $G_u(t_u)$ ($u \in (0, \ E)$) strictly decreases from $+\infty$ to 1. The equations $w_u(t_u) = 0$ and $G_u(t_u) = e^{rT} = e^{0.08}$ yield the price $u \doteqdot 17.8157$. With this price, if investors continue to invest $t_u \doteqdot 0.5434$ of their current capital, they can maximize the limit expectation of growth rate to $e^{0.08} \doteqdot 1.0833$.

In the prevailing pricing theory, $E / u = e^{rT}$ yields the price

$$u = Ke^{-rT} N\left(-\frac{\log\frac{S}{K} + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) - SN\left(-\frac{\log\frac{S}{K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right), (22)$$

which is the Black-Scholes formula for a European put option. Substituting the above-mentioned values for this formula, we obtain the (higher) price $u \fallingdotseq 21.2176$ (> 17.8157). With this price, if the investors continue to invest $t_u \fallingdotseq 0.2278$ of their current capital, they can maximize the limit expectation of growth rate to 1.0096 (< 1.0833).

REMARK

This paper is a revised form of the pre-print "Game pricing and double sequence of random variables".

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